The Traveling Salesman Problem

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Definition: The **Traveling Salesman Problem (TSP)** is the problem of finding a **minimum-weight Hamilton circuit** in K_N.

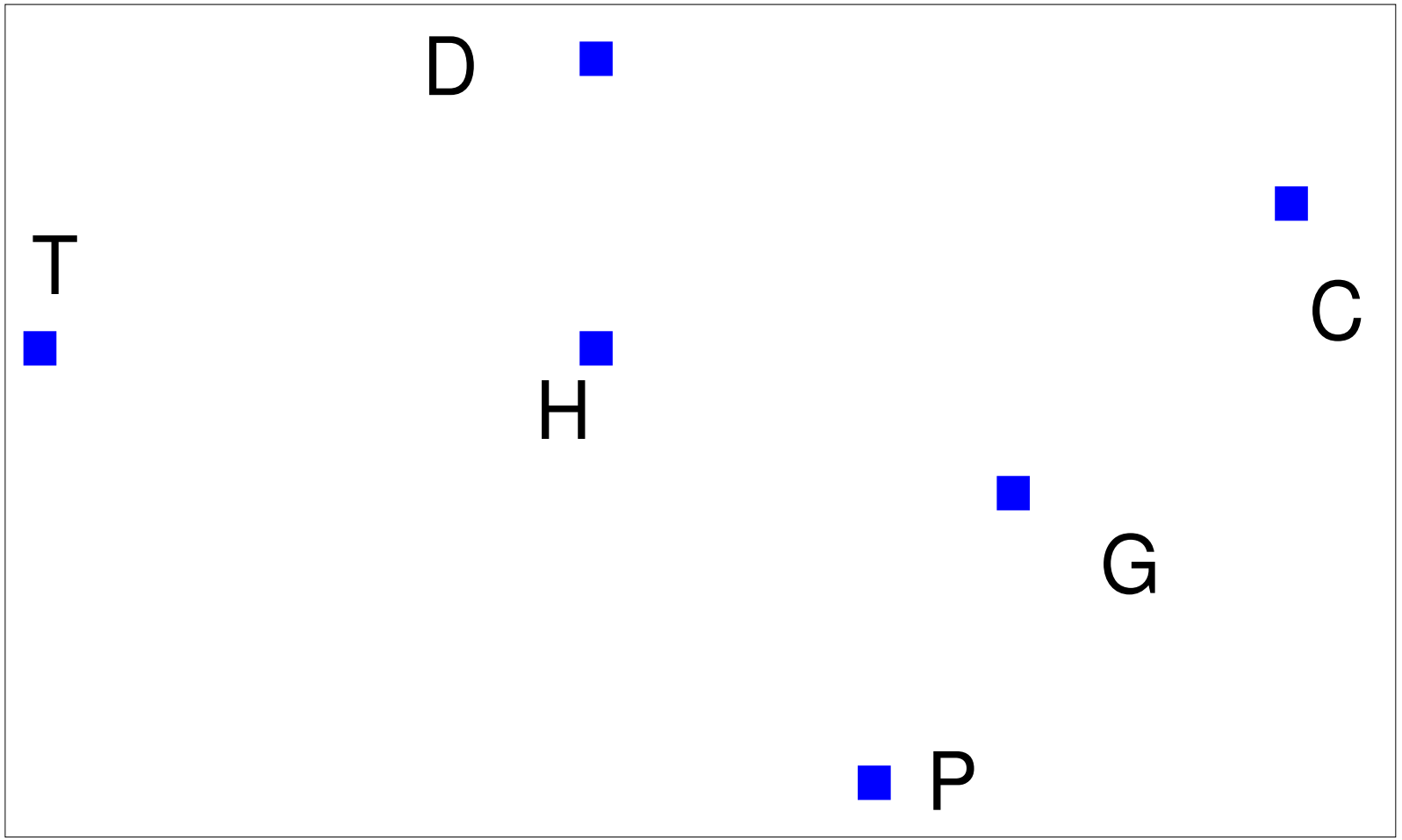
The Traveling Saleswitch Problem

Example: : Sabrina has the following list of errands:

- Pet store (the black cat needs a new litterbox) (P)
- Greenhouse (replenish supply of deadly nightshade) (G)
- Pick up black dress from cleaners (C)
- Drugstore (eye of newt, wing of bat, toothpaste) (D)
- Target (weekly special on cauldrons) (T)

In ~~witch~~ which order should she do these errands in order to minimize the time spent on her broom?

The Traveling Saleswitch Problem



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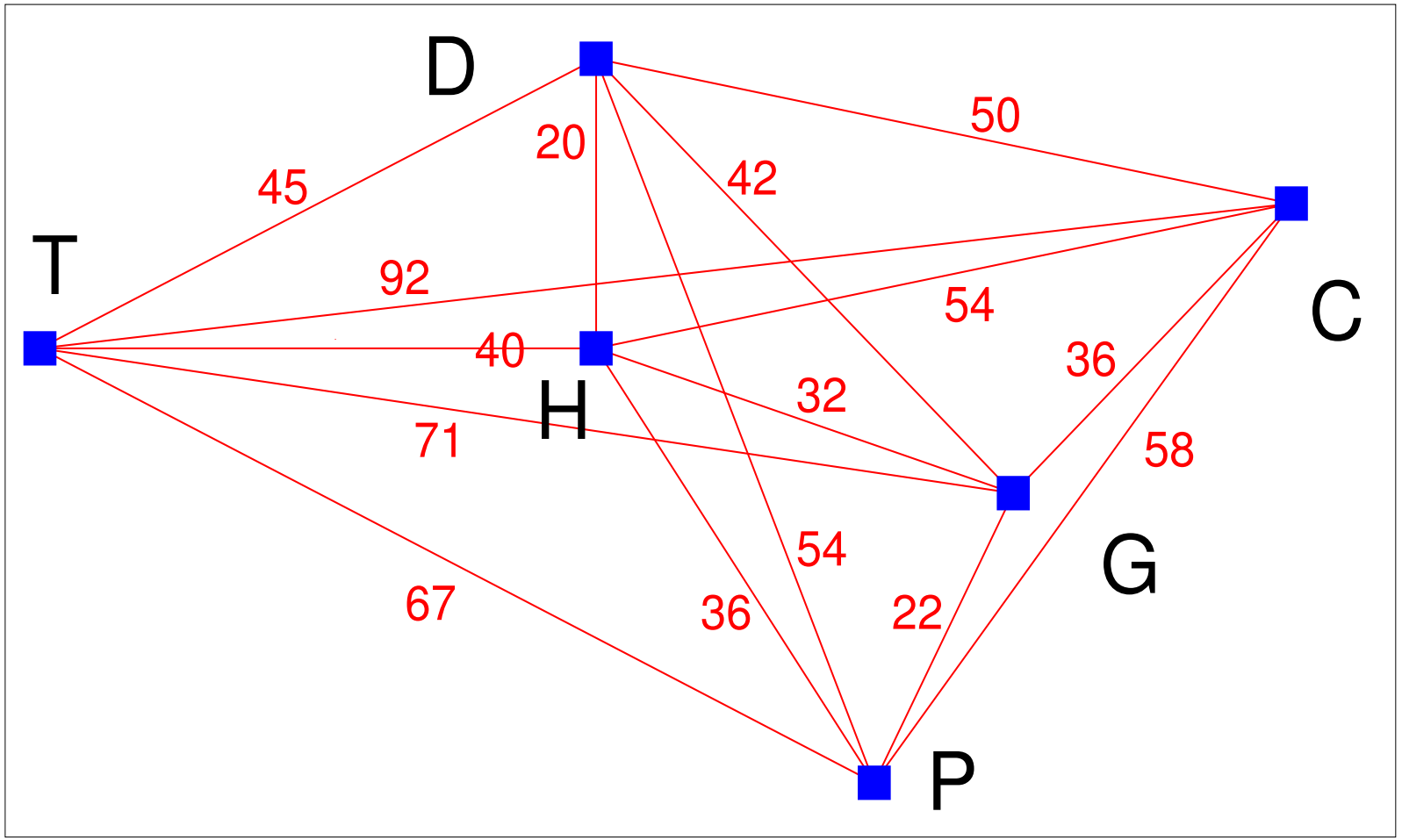

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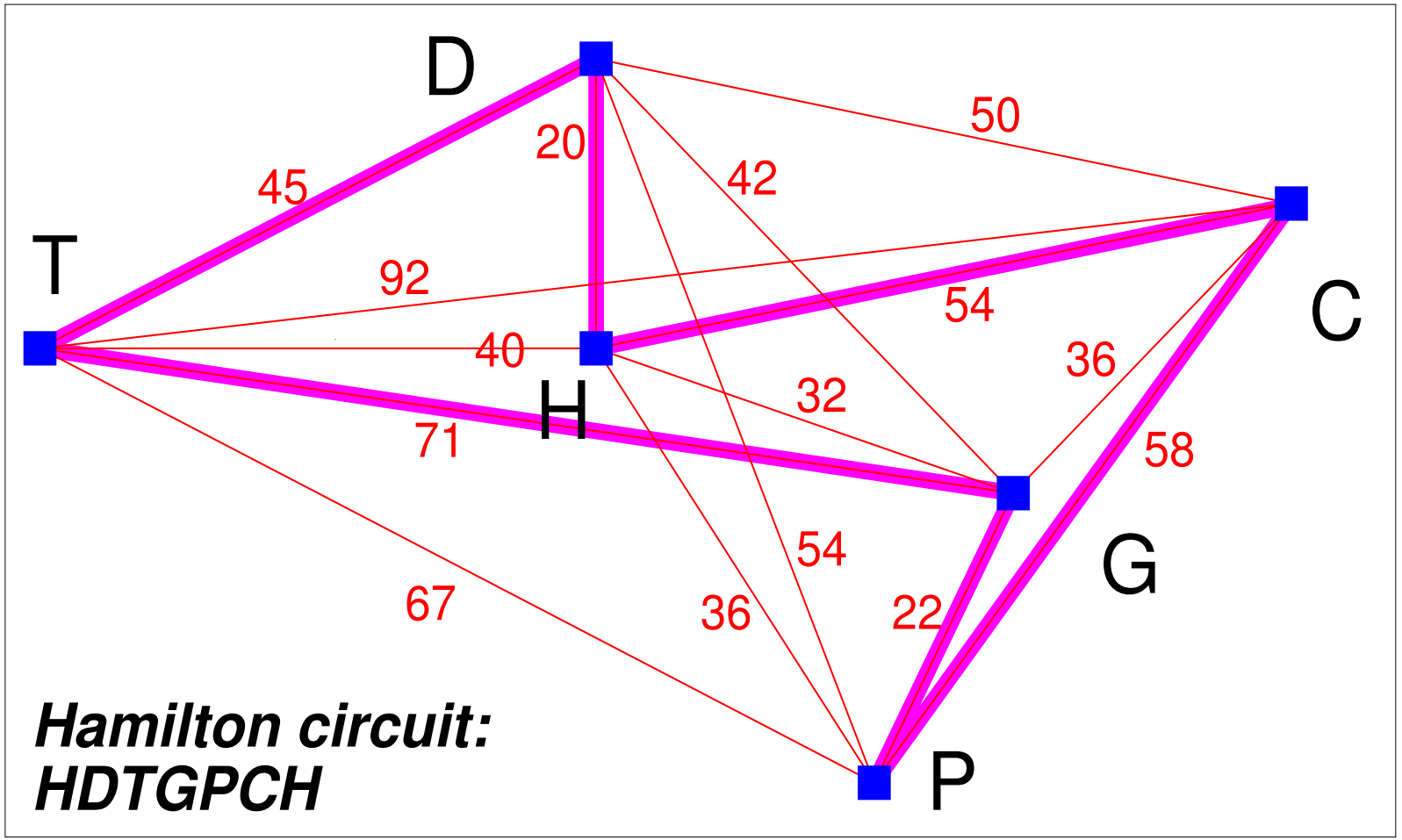

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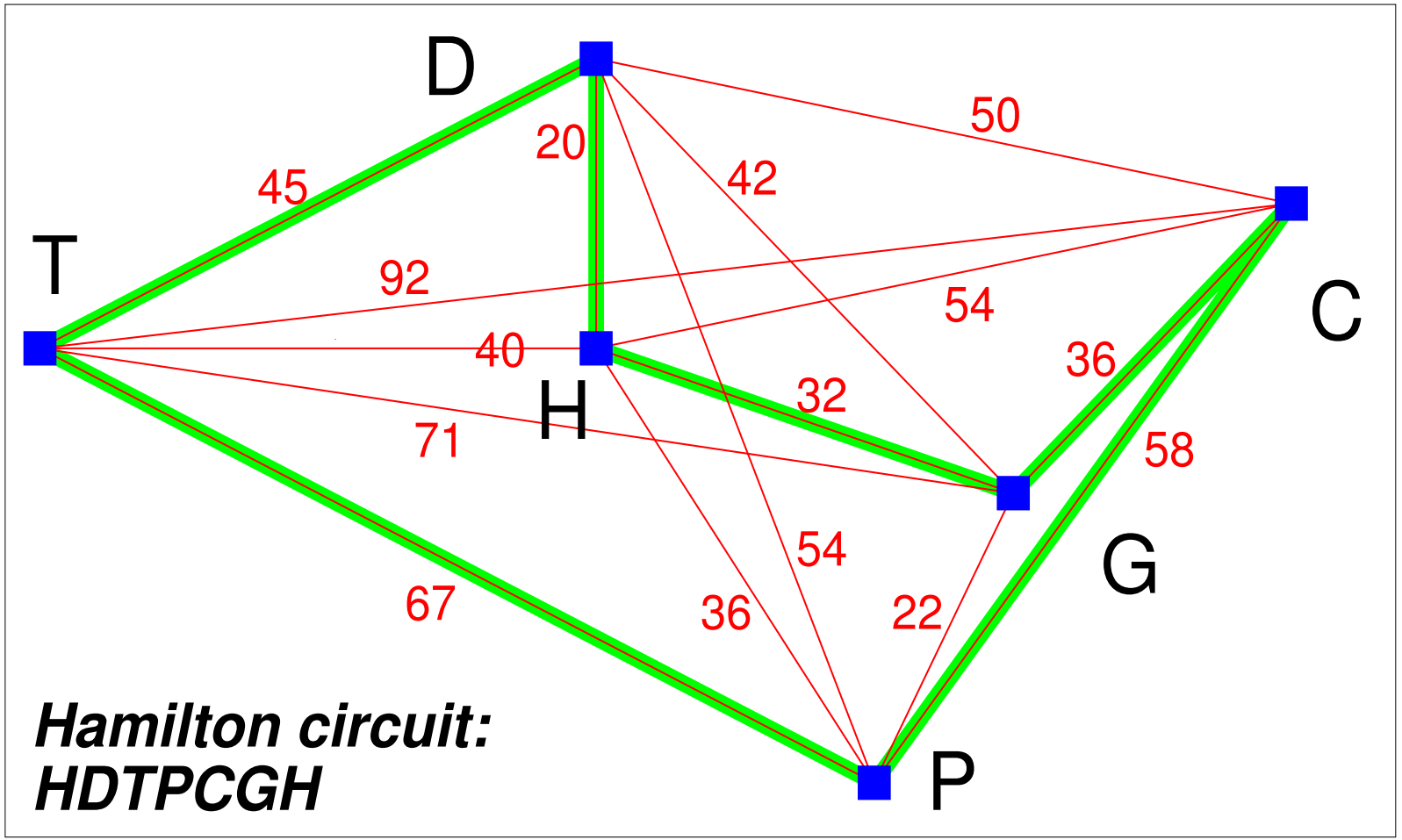

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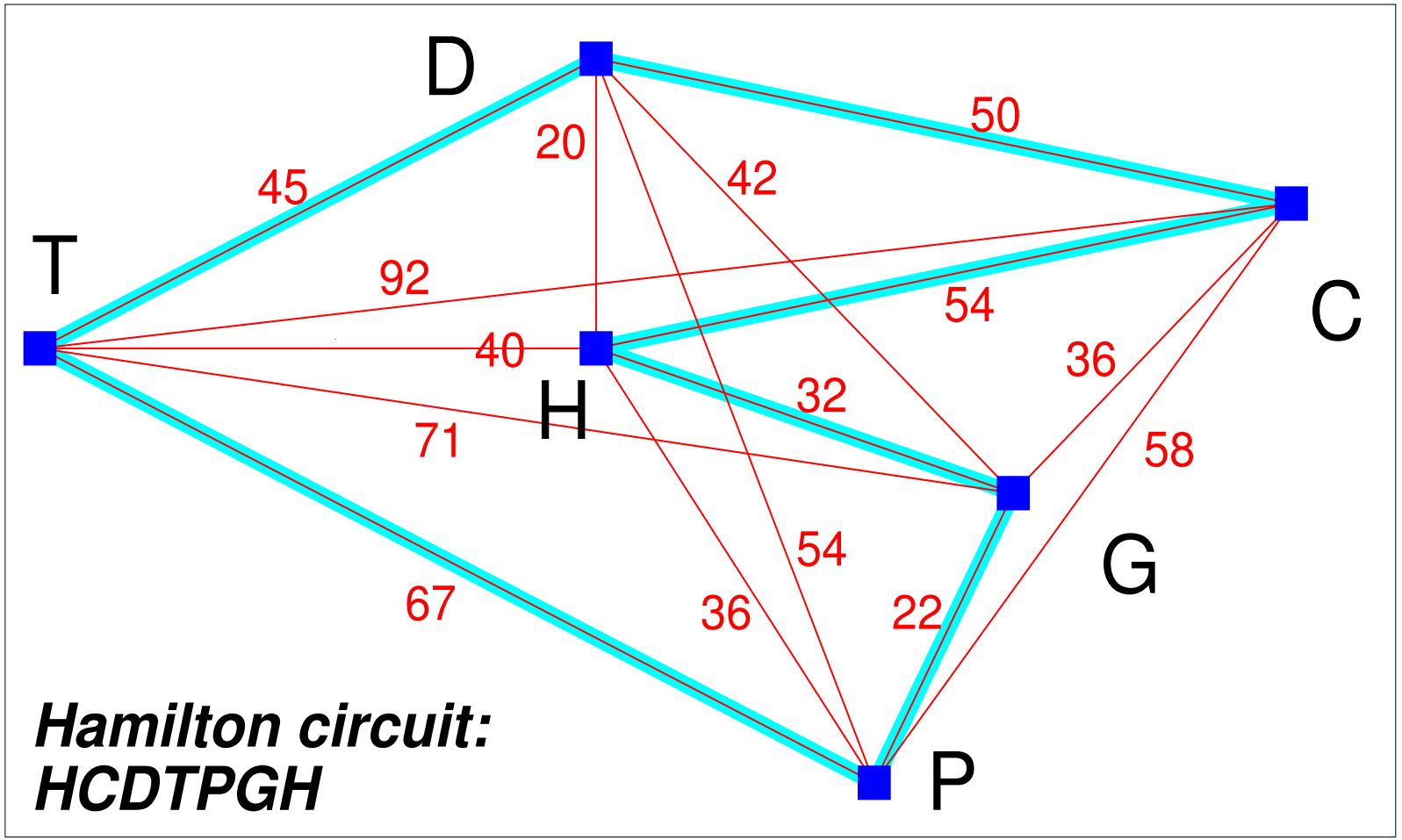

The Traveling Saleswitch Problem

Times between each pair of locations (minutes):

	H	P	G	C	D	T
Home (H)	0	36	32	54	20	40
Pet store (P)	36	0	22	58	54	67
Greenhouse (G)	32	22	0	36	42	71
Cleaners (C)	54	58	36	0	50	92
Drugstore (D)	20	54	42	50	0	45
Target (T)	40	67	71	92	45	0

Possible Hamilton Circuits

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$$\text{Weight(HDTGPCH)} = 20 + 45 + 71 + 22 + 58 + 54 = \mathbf{270}$$

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Possible Hamilton Circuits

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- ...the number of Hamilton circuits is

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

.

- How about listing all possible circuits?

Possible Hamilton Circuits (Page 1)

Hamilton circuit	Weight	Hamilton circuit	Weight
H,C,D,G,P,T,H	275	H,C,P,D,G,T,H	319
H,C,D,G,T,P,H	320	H,C,P,D,T,G,H	314
H,C,D,P,G,T,H	291	H,C,P,G,D,T,H	261
H,C,D,P,T,G,H	328	H,C,P,G,T,D,H	270
H,C,D,T,G,P,H	278	H,C,P,T,D,G,H	298
H,C,D,T,P,G,H	270	H,C,P,T,G,D,H	312
H,C,G,D,P,T,H	293	H,C,T,D,G,P,H	291
H,C,G,D,T,P,H	280	H,C,T,D,P,G,H	299
H,C,G,P,D,T,H	251	H,C,T,G,D,P,H	349
H,C,G,P,T,D,H	244	H,C,T,G,P,D,H	313
H,C,G,T,D,P,H	296	H,C,T,P,D,G,H	341
H,C,G,T,P,D,H	302	H,C,T,P,G,D,H	297

Possible Hamilton Circuits (Page 2)

Hamilton circuit	Weight	Hamilton circuit	Weight
H,D,C,G,P,T,H	235	H,D,P,C,G,T,H	279
H,D,C,G,T,P,H	280	H,D,P,C,T,G,H	327
H,D,C,P,G,T,H	261	H,D,P,G,C,T,H	264
H,D,C,P,T,G,H	298	H,D,P,G,T,C,H	313
H,D,C,T,G,P,H	291	H,D,P,T,C,G,H	301
H,D,C,T,P,G,H	283	H,D,P,T,G,C,H	302
H,D,G,C,P,T,H	263	H,D,T,C,G,P,H	251
H,D,G,C,T,P,H	293	H,D,T,C,P,G,H	269
H,D,G,P,C,T,H	274	H,D,T,G,C,P,H	266
H,D,G,P,T,C,H	297	H,D,T,G,P,C,H	270
H,D,G,T,C,P,H	319	H,D,T,P,C,G,H	258
H,D,G,T,P,C,H	312	H,D,T,P,G,C,H	244

Possible Hamilton Circuits (Page 3)

Hamilton circuit	Weight	Hamilton circuit	Weight
H,G,C,D,P,T,H	279	H,G,P,C,D,T,H	247
H,G,C,D,T,P,H	266	H,G,P,C,T,D,H	269
H,G,C,P,D,T,H	265	H,G,P,D,C,T,H	290
H,G,C,P,T,D,H	258	H,G,P,D,T,C,H	299
H,G,C,T,D,P,H	295	H,G,P,T,C,D,H	283
H,G,C,T,P,D,H	301	H,G,P,T,D,C,H	270
H,G,D,C,P,T,H	289	H,G,T,C,D,P,H	335
H,G,D,C,T,P,H	319	H,G,T,C,P,D,H	327
H,G,D,P,C,T,H	318	H,G,T,D,C,P,H	292
H,G,D,P,T,C,H	341	H,G,T,D,P,C,H	314
H,G,D,T,C,P,H	305	H,G,T,P,C,D,H	298
H,G,D,T,P,C,H	298	H,G,T,P,D,C,H	328

Possible Hamilton Circuits (Page 4)

Hamilton circuit	Weight	Hamilton circuit	Weight
H,P,C,D,G,T,H	297	H,P,G,C,D,T,H	229
H,P,C,D,T,G,H	292	H,P,G,C,T,D,H	251
H,P,C,G,D,T,H	257	H,P,G,D,C,T,H	282
H,P,C,G,T,D,H	266	H,P,G,D,T,C,H	291
H,P,C,T,D,G,H	305	H,P,G,T,C,D,H	291
H,P,C,T,G,D,H	319	H,P,G,T,D,C,H	278
H,P,D,C,G,T,H	287	H,P,T,C,D,G,H	319
H,P,D,C,T,G,H	335	H,P,T,C,G,D,H	293
H,P,D,G,C,T,H	300	H,P,T,D,C,G,H	266
H,P,D,G,T,C,H	349	H,P,T,D,G,C,H	280
H,P,D,T,C,G,H	295	H,P,T,G,C,D,H	280
H,P,D,T,G,C,H	296	H,P,T,G,D,C,H	320

Possible Hamilton Circuits (Page 5)

Hamilton circuit	Weight	Hamilton circuit	Weight
H,T,C,D,G,P,H	282	H,T,G,C,D,P,H	287
H,T,C,D,P,G,H	290	H,T,G,C,P,D,H	279
H,T,C,G,D,P,H	300	H,T,G,D,C,P,H	297
H,T,C,G,P,D,H	264	H,T,G,D,P,C,H	319
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H,T,C,P,G,D,H	274	H,T,G,P,D,C,H	291
H,T,D,C,G,P,H	229	H,T,P,C,D,G,H	289
H,T,D,C,P,G,H	247	H,T,P,C,G,D,H	263
H,T,D,G,C,P,H	257	H,T,P,D,C,G,H	279
H,T,D,G,P,C,H	261	H,T,P,D,G,C,H	293
H,T,D,P,C,G,H	265	H,T,P,G,C,D,H	235
H,T,D,P,G,C,H	251	H,T,P,G,D,C,H	275

Solving the TSP by Brute Force

What we have just done is the **Brute-Force Algorithm**:

- Make a list of all possible Hamilton circuits
- Calculate the weight of each Hamilton circuit by adding up the weights of its edges.
- Choose the Hamilton circuit with the smallest total weight.

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- The Brute-Force Algorithm is **optimal**: it is guaranteed to find a solution.
- OTOH, the algorithm is **inefficient**: it has to look at all $(N-1)!$ Hamilton circuits, and this can take a long time.

Solving the TSP by Brute Force

If your computer can compute **one million** Hamilton circuits per second...

- $N = 6, 7, 8, 9$: instantaneous
- $N = 10$: about $1/3$ second
- $N = 11$: about 4 seconds
- $N = 12$: about 40 seconds
- $N = 13$: about 8 minutes
- $N = 14$: nearly 2 hours
- $N = 15$: a little over a day
- $N = 20$: over a million years

Solving the TSP Without Brute Force

Is there a better way to tackle the TSP?

That is, is there an optimal algorithm that is also efficient?

Solving the TSP Without Brute Force

Idea: At each stage in your tour, choose the closest vertex that you have not visited yet.

This is called the **Nearest-Neighbor Algorithm.**

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- If Sabrina starts at home, the closest destination is the drugstore.
- So, perhaps the Hamilton circuit ought to begin H,D.

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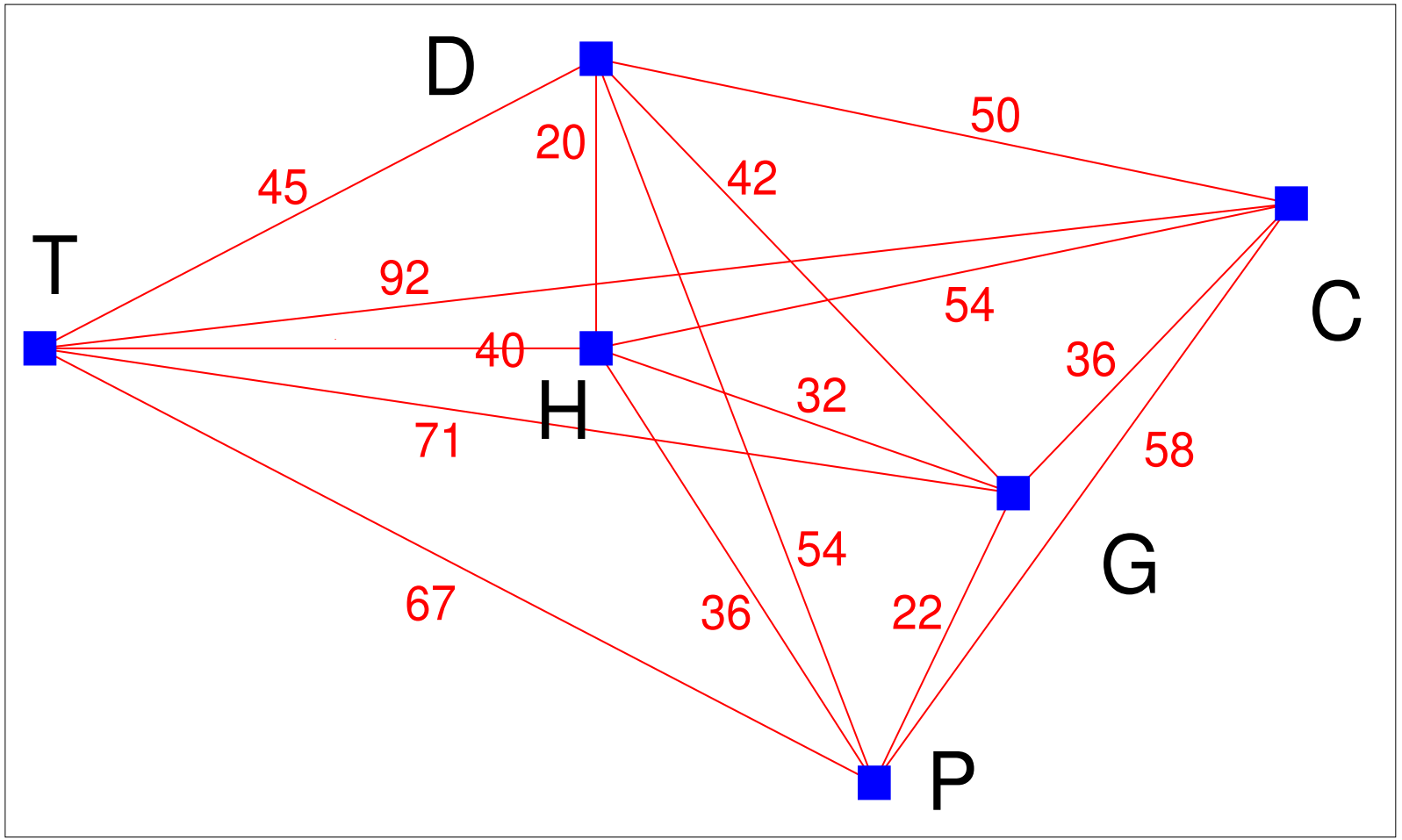

The Traveling Saleswitch Problem

Eventually, we end up with the Hamilton circuit

H, T, C, P, G, D, H.

- Weight of this circuit: 274
- Weight of an optimal circuit: 229
- Average weight of a circuit: 287.6

Comparing Brute-Force and Nearest-Neighbor

The Brute-Force Algorithm is **optimal** but **inefficient**.

- It is guaranteed to find a solution, but it may take an unreasonably long time to do so.

The Nearest-Neighbor Algorithm is **efficient** but **nonoptimal**.

- It is quick and easy, but does not always find the lowest-weight Hamilton circuit.