The Mathematics of Touring (Chapter 6)

In Chapter 5, we studied Euler paths and Euler circuits: paths and circuits that use every **edge** of a graph.

What if the goal is to visit every **vertex** instead of every edge?

The Traveling Salesman Problem (TSP)

Willy, a traveling salesman, has to visit each of several cities (say, the 48 state capitals of the continental United States)

He would like his trip to cover as little distance as possible.

In what order should Willy visit the 48 cities?

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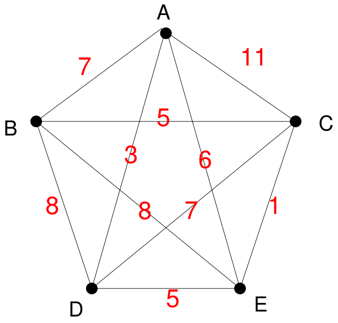
- A spacecraft needs to visit each of six sites on Mars to collect samples (fuel is very expensive on Mars!)
- A school bus needs to visit each of several pickup/dropoff locations (here the issue is not money, but time)
- A mother is taking her four-year-old trick-or-treating and needs to visit each of eight friends and relatives (student in this class 10 years ago)

The TSP As A Graph Problem

Suppose we have a graph in which every edge has a **weight** (representing its cost, time, or distance).



The TSP is then to find a path or a circuit that

- visits every vertex; and
- has total weight as low as possible.

Hamilton Paths and Hamilton Circuits

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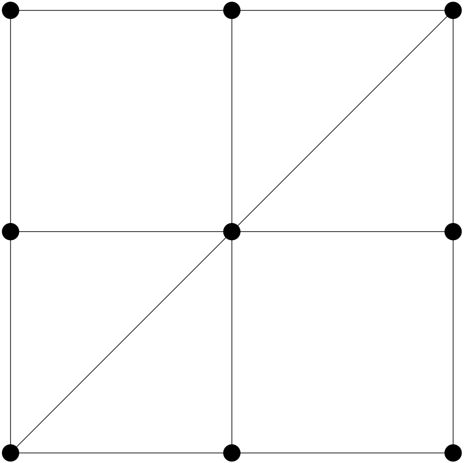
- By contrast, an **Euler path/circuit** is a path/circuit that uses every **edge** exactly once.
- Reminder: "Path" means that the starting and ending vertices are different; "circuit" means that they are the same.

Hamilton Paths and Hamilton Circuits



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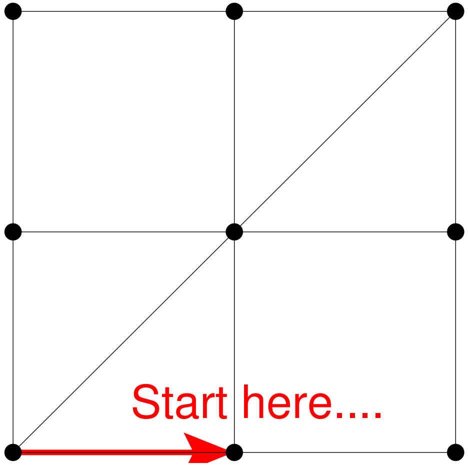

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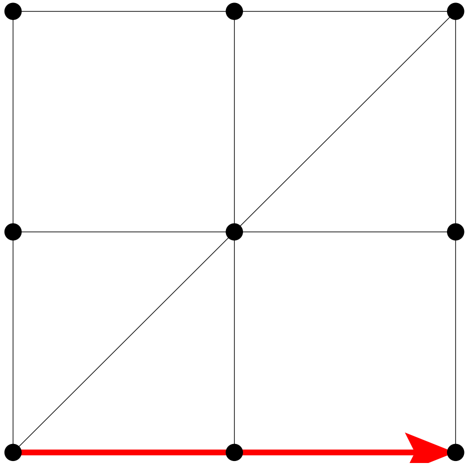

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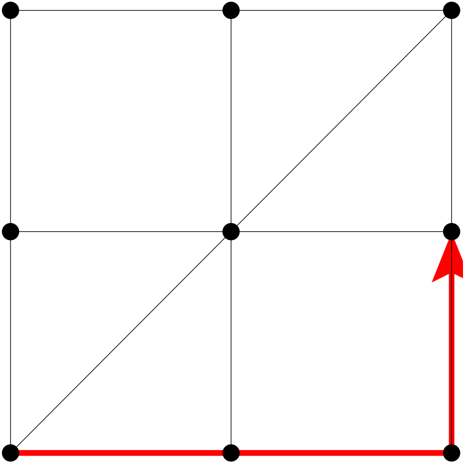

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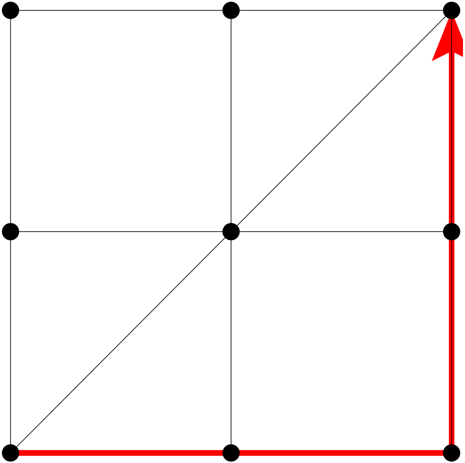

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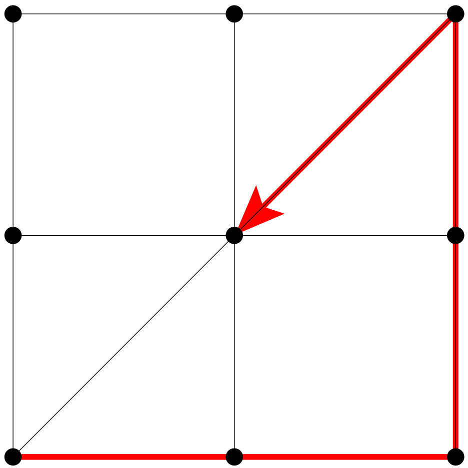

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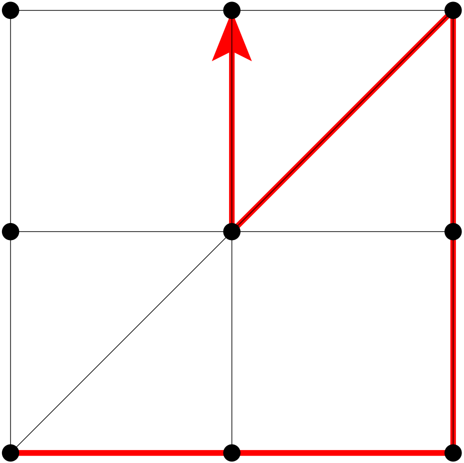

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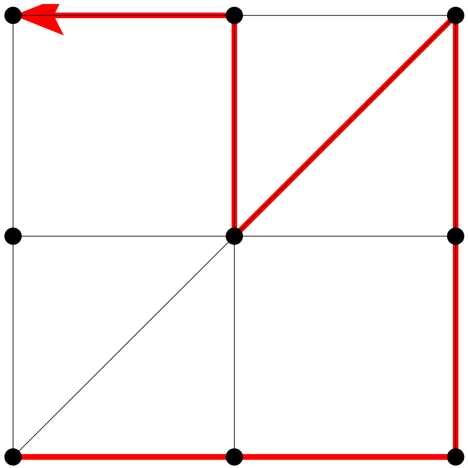

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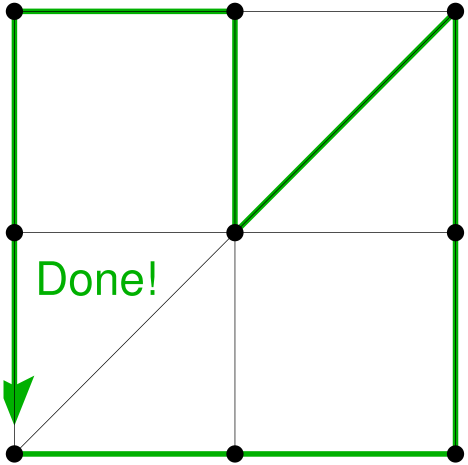
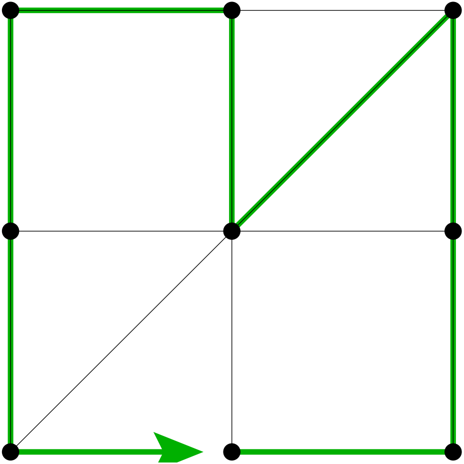


Hamilton Paths and Hamilton Circuits

Changing the starting vertex (or "reference vertex") does not change the Hamilton circuit, because the same edges are traversed in the same directions.

Hamilton Paths and Hamilton Circuits



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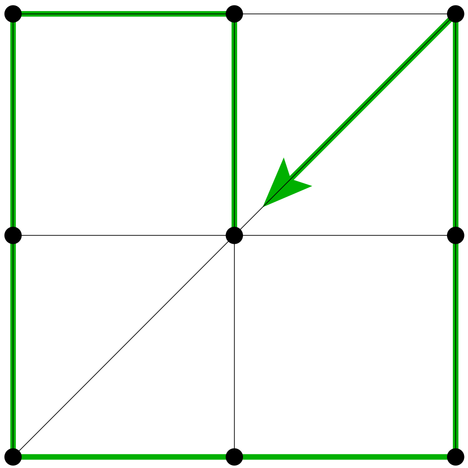
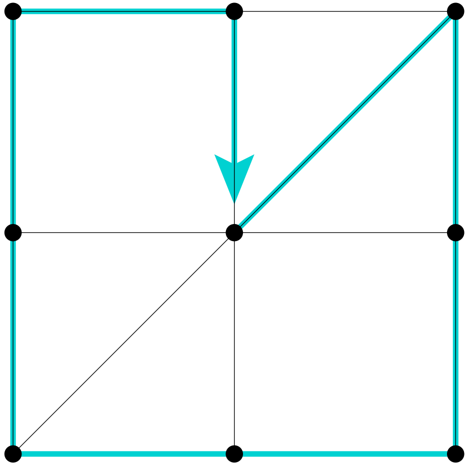


Hamilton Paths and Hamilton Circuits

We can also make a Hamilton circuit into its “mirror image” by reversing direction. The mirror image uses the same edges, but **backwards**, so it is not considered the same as the original Hamilton circuit.

Hamilton Paths and Hamilton Circuits



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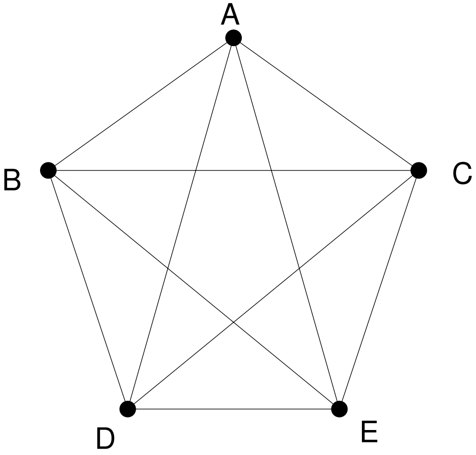

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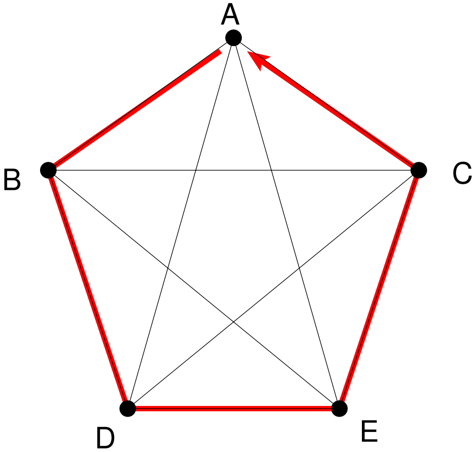

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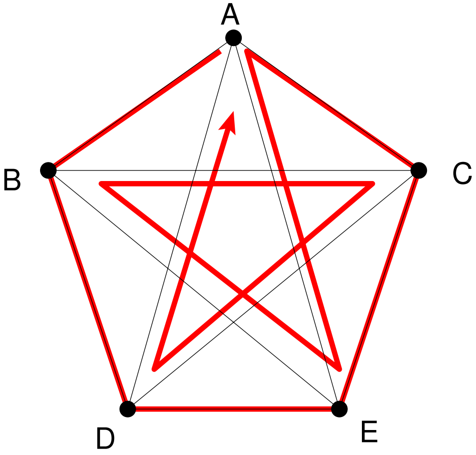

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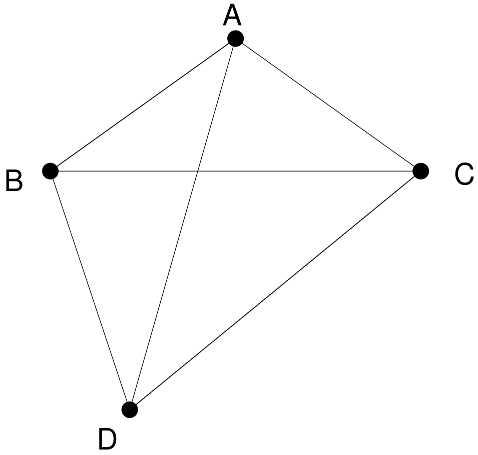

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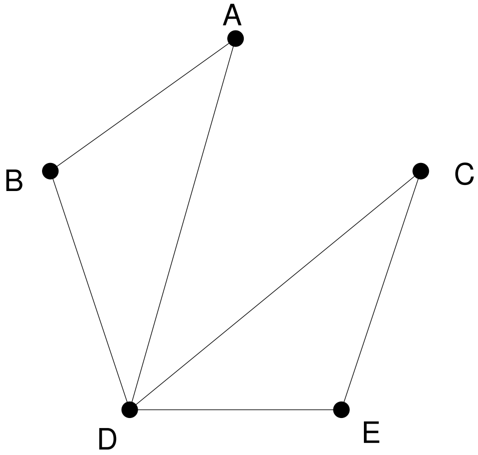

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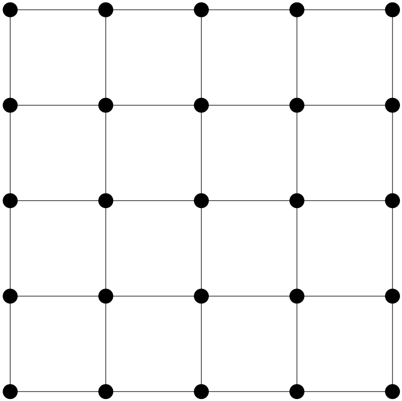

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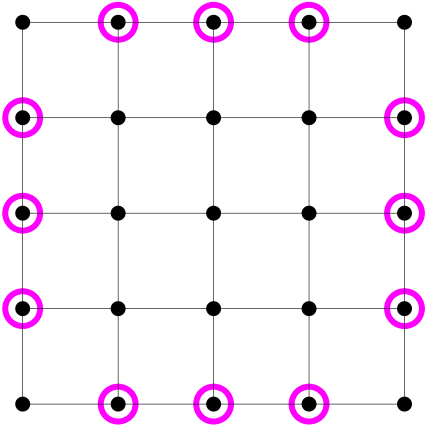

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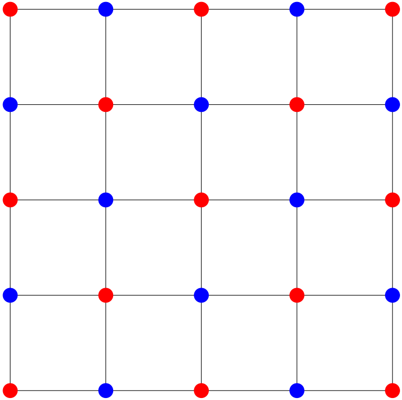

Hamilton vs. Euler

Conclusion: Whether a graph does or does not have a Hamilton circuit **tells you nothing** about whether it has an Euler circuit, and vice versa.

The same is true for Hamilton/Euler **paths** (rather than circuits).

Which Graphs Have Hamilton Circuits?

We know how to determine whether a graph has an Euler path or circuit: count the odd vertices.

On the other hand, **there is no simple way to tell whether or not a given graph has a Hamilton path or circuit**.

Finding the Shortest Hamilton Circuit

Rather than asking whether a particular graph has a Hamilton circuit, we will be looking at graphs with **lots** of Hamilton circuits, and trying to find the **shortest** one.

For example, Willy the traveling salesman has the option to drive from any state capital to any other, so the graph he lives in has lots of edges.