Discrete Fair-Division Problems

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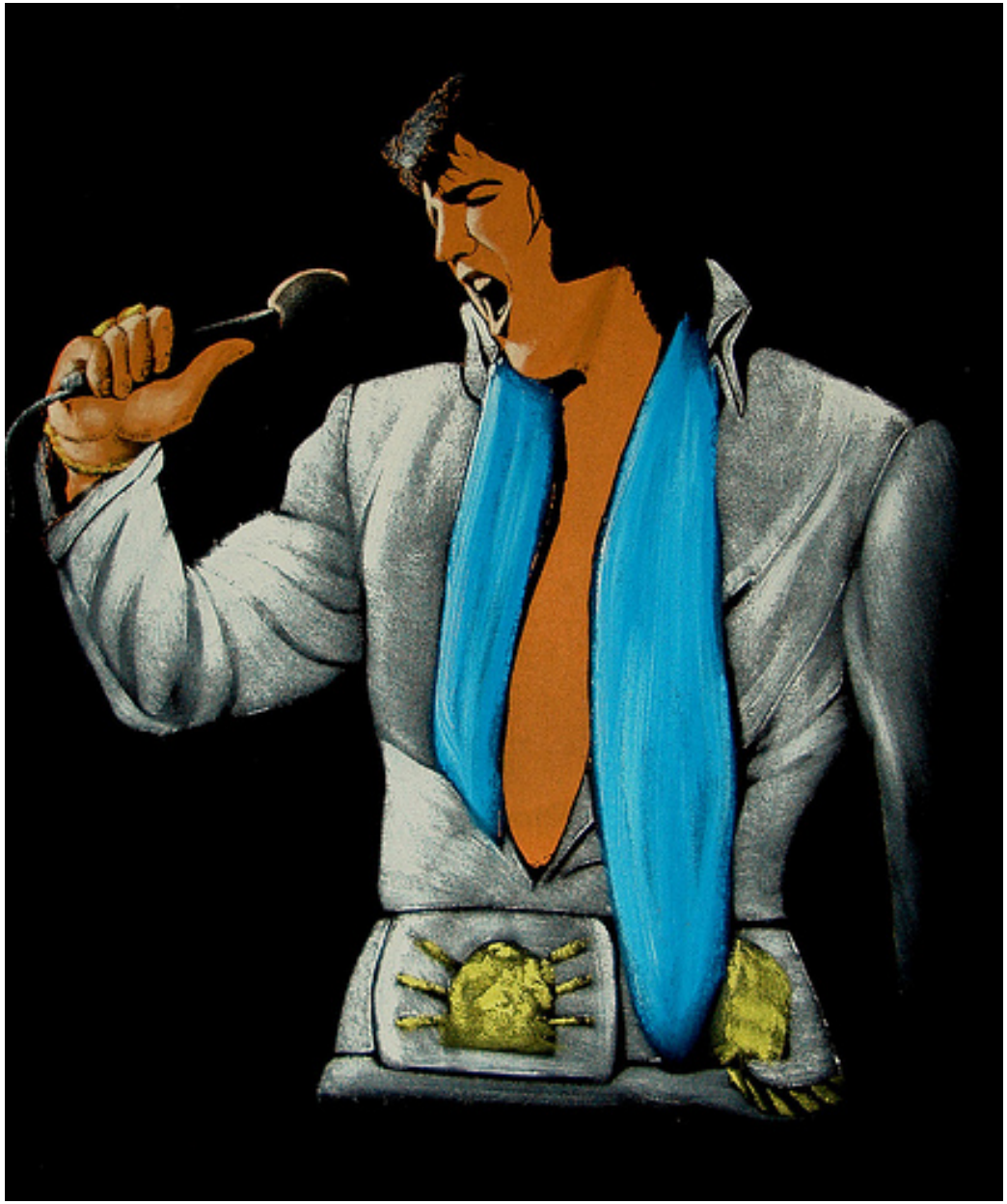
They (and their lawyers) are currently arguing over who gets the velvet Elvis.



Discrete Fair-Division Problems

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They (and their lawyers) are currently arguing over who gets the velvet Elvis.

Casey thinks it is worth \$1000 and Jordan thinks it is worth \$600.

How do you, as the divorce court judge, settle the matter? 

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How do you, as the divorce court judge, settle the matter? 

We'll come back to this.

The Method of Sealed Bids

- Fair-division method for a multiplayer game with **discrete** goods
- Best method for a game in which the goods have widely varying values
- Similar to an auction; may require players to put in cash

The Method of Sealed Bids

1. **Bidding.** Each player determines how much each item is worth, then submits bids on each item.
   No player gets to see the other players' bids until all bids are submitted.

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3. **First Settlement.** Each player then either pays or receives money to make sure he/she has a fair share.

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3. **First Settlement.** Each player then either pays or receives money to make sure he/she has a fair share.

4. **Division of Surplus.** There is usually money left over, which can be divided equally among the players.

The Method of Sealed Bids: An Example

Four squabbling siblings (Pauline, Quentin, Roberta, and Severus) are joint heirs to an estate consisting of a castle in Spain, a 75-foot sailboat, and a replica of the Statue of Liberty.

Step 1: Bidding. Each player determines how much each item is worth and submits bids on each item.

(To maintain the privacy assumption, each player can seal his/her bids in an envelope, then all envelopes are opened simultaneously.)

The Method of Sealed Bids: An Example

	Pauline	**Quentin**	**Roberta**	**Severus**
Castle				
Boat				
Statue				
Total				
Fair Share				

(All amounts in \$1000s.)

The Method of Sealed Bids: An Example

	Pauline	**Quentin**	**Roberta**	**Severus**
Castle	200	250	180	210
Boat	50	30	40	40
Statue	6	20	4	10
Total				
Fair Share				

(All amounts in $1000s.)

The Method of Sealed Bids: An Example

	Pauline	**Quentin**	**Roberta**	**Severus**
Castle	200	250	180	210
Boat	50	30	40	40
Statue	6	20	4	10
Total	256	300	224	260
Fair Share				

(All amounts in \$1000s.)

The Method of Sealed Bids: An Example

	Pauline	**Quentin**	**Roberta**	**Severus**
Castle	200	250	180	210
Boat	50	30	40	40
Statue	6	20	4	10
Total	256	300	224	260
Fair Share	64	75	56	65

(All amounts in $1000s.)

The Method of Sealed Bids: An Example

Step 2: Allocation. Each item goes to the highest bidder. (If there is a tie for some item, choose randomly.)

	Pauline	**Quentin**	**Roberta**	**Severus**
Castle	200	**250**	180	210
Boat	**50**	30	40	40
Statue	6	**20**	4	10
Total	256	300	224	260
Fair Share	64	75	56	65
Item Value	**50**	**270**	**0**	**0**

The Method of Sealed Bids: An Example

Step 3: First Settlement. For each player, compare the total value of the items she received to her fair share, to obtain the amount of money she owes to, or is owed by, the estate.

	Pauline	**Quentin**	**Roberta**	**Severus**
Castle	200	**250**	180	210
Boat	**50**	30	40	40
Statue	6	**20**	4	10
Total	256	300	224	260
Fair Share	64	75	56	65
Item Value	50	270	0	0
Settlement	14	-195	56	65

The Method of Sealed Bids: An Example

Summary:

- Pauline receives the sailboat and $14,000 in cash.
- Quentin receives the castle and the statue, but has to pay the estate $195,000.
- Roberta receives $56,000 from the estate.
- Severus receives $65,000 from the estate.

The Method of Sealed Bids: An Example

Summary:

- Pauline receives the sailboat and $14,000 in cash.
- Quentin receives the castle and the statue, but has to pay the estate $195,000.
- Roberta receives $56,000 from the estate.
- Severus receives $65,000 from the estate.

We're not done yet.

The Method of Sealed Bids: An Example

Step 4: Division of Surplus.

- At this point, there is excess money!
- The estate has **$60,000** left over
  $(60 = 195 - 14 - 56 - 65)$.
- This is the **surplus**. Split it equally between the four players.

The Method of Sealed Bids: An Example

Step 5: Final Settlement.

	Pauline	**Quentin**	**Roberta**	**Severus**
Castle	200	**250**	180	210
Boat	**50**	30	40	40
Statue	6	**20**	4	10
Total	256	300	224	260
Fair Share	64	75	56	65
Item Value	50	270	0	0
First Settlement	14	-195	56	65
$\frac{1}{4}$ of \$60 surplus	15	15	15	15
Final Settlement	29	-180	71	80

The Method of Sealed Bids: An Example

Summary

	Pauline	**Quentin**	**Roberta**	**Severus**
Items	Boat	Castle, statue	—	—
Fair Share	64	75	56	65
Item Value	50	270	0	0
Cash Value	29	−180	71	80
Share Value	**79**	**90**	**71**	**80**

- Every player receives a share that is \$15,000 **more** than what he or she considers to be a fair share!

The Method of Sealed Bids: Example #2

Five students find a treasure chest buried deep in the bowels of Snow Hall. The chest contains

- a coupon entitling the bearer to an A in a mathematics course of his or her choosing;

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- a ticket to the KU Quidditch Team's upcoming match;

The Method of Sealed Bids: Example #2

Five students find a treasure chest buried deep in the bowels of Snow Hall. The chest contains

- a coupon entitling the bearer to an A in a mathematics course of his or her choosing;
- a ticket to the KU Quidditch Team's upcoming match;
- and a voodoo doll that looks oddly like Prof. Martin, complete with pins stuck in it.

The Method of Sealed Bids: Example #2

Here are the bids made in class on 10/5/11.

	Matt	**Shawnee**	**Nick**	**Mack**	**Sara**
Coupon	\$250	\$100	\$200	\$500	\$250,000,000
Ticket	\$72	\$5	\$1	\$10	\$10
Doll	\$6	\$10	\$150	\$50	\$2

The Method of Sealed Bids: Example #2

- Each item goes to the **highest bidder**.
- Add up each player's valuations of the three items to get his or her **total value** for the booty.
- Divide by 5 (the number of players) to get his or her estimate of a **fair share.**
- Find the value of the **item(s)** allocated to each player.

The Method of Sealed Bids: Example #2

	Matt	Shawnee	Nick	Mack	Sara
Coupon	$250	$100	$200	$500	$250,000,000
Ticket	$72	$5	$1	$10	$10
Doll	$6	$10	$150	$50	$2
Total Value	$328	$115	$351	$560	$250,000 012.00
Fair Share	$65.60	$23.00	$70.20	$112.00	$50,000, 002.40
Item Value	$72	$0	$150	$0	$250,000, 000.00

The Method of Sealed Bids: Example #2

- Determine the **first settlement** by subtracting each player's item value from his/her fair share. (If negative, the player must pay money; if positive, the player receives money.)

The Method of Sealed Bids: Example #2

	Matt	**Shawnee**	**Nick**	**Mack**	**Sara**
Items	Ticket	—	Doll	—	Coupon
Total Value	\$328	\$115	\$351	\$560	\$250,000 012.00
Fair Share	\$65.60	\$23.00	\$70.20	\$112.00	\$50,000, 002.40
Item Value	\$72	\$0	\$150	\$0	\$250,000, 000.00
First Sett't	−\$6.40	\$23.00	−\$79.80	\$112.00	−\$199, 999, 997.60

The Method of Sealed Bids: Example #2

- The net amount of money received by the players in the first settlement is

$$\$(-6.40 + 23.00 - 79.80 + 112.00 - 199,999,997.60)$$
$$= \quad -\$\mathbf{199,999,948.80}.$$

  This number[1] (without the minus sign) is the **surplus**. Remember that the surplus is never negative!

- Divide the surplus into five equal shares of $39,999,989.76 each. Give each player one of the shares.
- Add this to the first settlement to produce the **final settlement**.

[1] The number calculated in class was $199,999,997.60.

The Method of Sealed Bids: Example #2

Player	Final Settlement
Matt	$39,999,983.36
Shawnee	$40,000,012.76
Nick	$39,999,909.96
Mack	$40,000,101.76
Sara	-$160,000,007.84

Notice that these numbers add up to 0.

The Method of Sealed Bids: Example #2

Analysis of this example (as discussed in class):

- What if Sara was allowed to change her bid on the coupon to \$501?
- What if, after Sara changed her bid, Mack was then allowed to change his bid to \$502?
- If Nick had known that Sara was going to bid \$250,000,000 for the coupon, what would he have bid for it instead of \$200?
- All these thought experiments demonstrate why the Privacy Assumption is vital!

The Method of Sealed Bids: Overview

Advantage: The Method of Sealed Bids works well when the booty to be divided consists of only a few items of widely differing values (e.g., an estate).

Other methods, such as the Lone-Divider Method, do not work well in this situation, because it may be impossible for the divider to separate the booty into two or more equal shares.

The Method of Sealed Bids: Overview

Advantage: The Method of Sealed Bids encourages honesty!

- If you underbid, your total valuation may end up too low and you may not receive your fair share.
- If you overbid, you may end up paying a lot of money for something you don't want.
- The Privacy Assumption is vital. It would be a big advantage to know the other players' bids.

The Method of Sealed Bids: Overview

Disadvantage: The Method of Sealed Bids requires players to have enough cash to "cover their bids".

This assumption may not always hold in practice.