Sequential Coalitions

There is another approach to measuring power, due to the mathematicians Shapley and Shubik (in fact, in 1954, predating Banzhaf's 1965 work).

Idea: Instead of regarding coalitions as groups of players who join **all at once**,

think of coalitions as groups that players join **one at a time**.

That is, we are looking not at coalitions, but at

sequential coalitions.

Sequential Coalitions and Pivotal Players

Example 1: In the WVS [17; 6, 4, 3, 3, 2, 1, 1] ...

- First, P_4 proposes a motion.
- Then, P_6 agrees to vote yes.
- Then, P_1 agrees to vote yes.
- Then, P_5 agrees.
- Then, P_2 agrees.
- Then, P_3 agrees.
- Then, P_7 agrees.

Who is the pivotal player?

Sequential Coalitions and Pivotal Players

Example 1: In the WVS [17; 6, 4, 3, 3, 2, 1, 1] ...

- First, P_4 proposes a motion. (Vote tally: 3)
- Then, P_6 agrees to vote yes. (Vote tally: $3+1 = 4$)
- Then, P_1 agrees to vote yes. ($4+6 = 10$)
- Then, P_5 agrees. ($10+2 = 12$)
- Then, P_2 agrees. ($12+4 = 16$)
- Then, P_3 agrees. ($16+3 = 19$)
- Then, P_7 agrees. ($19+1 = 20$)

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- Then, P_3 agrees. ($16+3=19$)
- Then, P_7 agrees. ($19+1=20$)

The pivotal player is $\mathbf{P_3}$.

Sequential Coalitions and Pivotal Players

Example 2: In the WVS $[\mathbf{17}; 6, 4, 3, 3, 2, 1, 1]$...

Suppose instead that the players join the sequential coalition in a different order:

$$\langle P_7,\ P_3,\ P_2,\ P_5,\ P_1,\ P_6,\ P_4\rangle.$$

Who is the pivotal player?

Sequential Coalitions and Pivotal Players

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$$\langle P_7,\ P_3,\ P_2,\ P_5,\ P_1,\ P_6,\ P_4\rangle.$$

Who is the pivotal player?

Player $\mathbf{P_i}$	P_7	P_3	P_2	P_5	P_1	P_6	P_4
Tally	1	4	8	10	16	17	20

Sequential Coalitions and Pivotal Players

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Tally	1	4	8	10	16	**17**	20

Sequential Coalitions and Pivotal Players

Example 3: [**13**; 6, 4, 3, 3, 2, 1, 1] (note change in q!)

As in Example 2, suppose again that the players join the sequential coalition in the order:

$$\langle P_7,\ P_3,\ P_2,\ P_5,\ P_1,\ P_6,\ P_4 \rangle.$$

Now who is the pivotal player?

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Sequential Coalitions and Pivotal Players

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Tally	1	4	8	10	**16**	17	20
					$\uparrow$		

Sequential Coalitions and Pivotal Players

Example 3: In the WVS $[\mathbf{11};\ 7,\ 5,\ 4]$, what are **all** the possible sequential coalitions and their pivotal players?

Sequential Coalitions and Pivotal Players

Example 3: In the WVS [**11**; 7, 5, 4], what are **all** the possible sequential coalitions and their pivotal players?

Sequential coalition	**Tallies**	**Pivotal player**
$\langle P_1, \underline{P_2,} P_3\rangle$	7, **12**, 16	P_2
$\langle P_1, \underline{P_3,} P_2\rangle$	7, **11**, 16	P_3
$\langle P_2, \underline{P_1,} P_3\rangle$	5, **12**, 16	P_1
$\langle P_2, P_3, \underline{P_1}\rangle$	5, 9, **16**	P_1
$\langle P_3, \underline{P_1,} P_2\rangle$	4, **11**, 16	P_1
$\langle P_3, P_2, \underline{P_1}\rangle$	4, 9, **16**	P_1

Coalitions vs. Sequential Coalitions

	Coalitions	Sequential Coalitions
How many players?	Any number	All N players
How many are critical or pivotal?	None, some, or all players may be critical	Exactly one pivotal player
Notation	$\{P_2, P_3, P_5, P_7\}$ Order doesn't matter	$\langle P_3, P_1, P_2 \rangle$ Order matters

The Shapley-Shubik Power Index

Idea: The more sequential coalitions for which player P_i is pivotal, the more power s/he wields.

Let $SS_i =$ number of sequential coalitions where P_i is pivotal.

The **Shapley-Shubik power index** of player P_i is the fraction

$$\sigma_i \;=\; \frac{SS_i}{\text{total number of sequential coalitions.}}$$

and the **Shapley-Shubik power distribution** of the entire WVS is the list

$$(\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_N)$$

Sequential Coalitions and Pivotal Players

Example 3: WVS $[\mathbf{11};\ 7,\ 5,\ 4]$.

Sequential coalitions

$\langle P_1,\ \underline{P_2,}\ P_3\rangle$

$\langle P_1,\ \underline{P_3,}\ P_2\rangle$

$\langle P_2,\ \underline{P_1,}\ P_3\rangle$

$\langle P_2,\ P_3,\ \underline{P_1}\rangle$

$\langle P_3,\ \underline{P_1,}\ P_2\rangle$

$\langle P_3,\ P_2,\ \underline{P_1}\rangle$

$SS_1 = 4;\qquad \sigma_1 = 4/6 = 2/3$

$SS_2 = 1;\qquad \sigma_2 = 1/6$

$SS_3 = 1;\qquad \sigma_3 = 1/6$

Shapley-Shubik power distribution: (2/3, 1/6, 1/6).

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How many possible sequential coalitions are there in a WVS with N players?

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  ... ...

- After the first K players have joined, there are $\mathbf{N-K}$ remaining players who could join $(K+1)$th.

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$\dots \quad \dots \quad \dots \quad \dots$

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  $\ldots \quad \ldots \quad \ldots \quad \ldots$

- After the first K players have joined, there are $\mathbf{N-K}$ remaining players who could join $(K+1)$th.

  $\ldots \quad \ldots \quad \ldots \quad \ldots$

- There are **2** remaining players who could join next-to-last.

How Many Sequential Coalitions?

How many possible sequential coalitions are there in a WVS with N players?

- There are **N** different players who could join first.
- After the first player has joined, there are $\mathbf{N-1}$ remaining players who could join second.
- After the first two players have joined, there are $\mathbf{N-2}$ remaining players who could join third.

$\dots \quad \dots \quad \dots \quad \dots$

- After the first K players have joined, there are $\mathbf{N-K}$ remaining players who could join $(K+1)$th.

$\dots \quad \dots \quad \dots \quad \dots$

- There are **2** remaining players who could join next-to-last.
- There is just **1** remaining player who could join last.

How Many Sequential Coalitions?

The number of sequential coalitions with N players is

$$N \times (N-1) \times (N-2) \times \cdots \times 2 \times 1.$$

For short, this is called **N!** (pronounced "N factorial").

N	**N!**
1	1
2	$2 \times 1 = 2$
3	$3 \times 2 \times 1 = 6$
4	$4 \times 3 \times 2 \times 1 = 24$
5	$5 \times 4 \times 3 \times 2 \times 1 = 120$

N	**N!**
6	720
7	5040
10	3628800
20	[19 digits]
100	[158 digits]

Factorials

To be exact,

100! = 93,326,215,443,944,152,681,699,238,856,266,700, 490,715,968,264,381,621,468,592,963,895,217,599,993,229,915, 608,941,463,976,156,518,286,253,697,920,827,223,758,251,185, 210,916,864,000,000,000,000,000,000,000,000.

By comparison,

$$\mathbf{2^{100}} = 1,267,650,600,228,229,401,496,703,205,376$$

and

$$\mathbf{100^2} = 10,000.$$

Calculating Shapley-Shubik Power Indices

For any weighted voting system with N players:

1. Write out all sequential coalitions (there are $N!$ of them).
2. For each one, determine the pivotal player.
3. Count the number of times that each player P_i is pivotal. Call this number SS_i.
4. The **Shapley-Shubik power index** of P_i is

$$\sigma_i = SS_i \,/\, N!$$

The **Shapley-Shubik power distribution** of the weighted voting system is the complete list

$$(\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_N).$$

Shapley-Shubik Power: Example 1

Example 1: The WVS $[13;\ 9,\ 7,\ 5]$

Shapley-Shubik Power: Example 1

Example 1: The WVS [13; 9, 7, 5]

Sequential coalition	**Weight tallies**	**Pivotal player**
$\langle P_1, P_2, P_3 \rangle$	9, **16**	P_2
$\langle P_1, P_3, P_2 \rangle$	9, **14**	P_3
$\langle P_2, P_1, P_3 \rangle$	7, **16**	P_1
$\langle P_2, P_3, P_1 \rangle$	7, 12, **21**	P_1
$\langle P_3, P_1, P_2 \rangle$	5, **14**	P_1
$\langle P_3, P_2, P_1 \rangle$	5, 12, **21**	P_1

Shapley-Shubik Power: Example 1

Example 1: The WVS [13; 9, 7, 5]

Sequential coalition	**Weight tallies**	**Pivotal player**
$\langle P_1, P_2, P_3 \rangle$	9, **16**	P_2
$\langle P_1, P_3, P_2 \rangle$	9, **14**	P_3
$\langle P_2, P_1, P_3 \rangle$	7, **16**	P_1
$\langle P_2, P_3, P_1 \rangle$	7, 12, **21**	P_1
$\langle P_3, P_1, P_2 \rangle$	5, **14**	P_1
$\langle P_3, P_2, P_1 \rangle$	5, 12, **21**	P_1

$SS_1 = 4 \quad SS_2 = 1 \quad SS_3 = 1$

Shapley-Shubik power distribution: (4/6, 1/6, 1/6)

Shapley-Shubik Power: Example 2

Example 2: The weighted voting system [8; 5, 5, 4].

Sequential coalition	Weight tallies	Pivotal player
$\langle P_1, P_2, P_3 \rangle$	5, **10**	P_2
$\langle P_1, P_3, P_2 \rangle$	5, **9**	P_3
$\langle P_2, P_1, P_3 \rangle$	5, **10**	P_1
$\langle P_2, P_3, P_1 \rangle$	5, **9**	P_3
$\langle P_3, P_1, P_2 \rangle$	4, **9**	P_1
$\langle P_3, P_2, P_1 \rangle$	4, **9**	P_2

$SS_1 = 2 \quad SS_2 = 2 \quad SS_3 = 2$

Sh.-Sh. power distribution: $(2/6, 2/6, 2/6) = (1/3, 1/3, 1/3)$

Shapley-Shubik Power: Example 2

Example 2: The weighted voting system [8; 5, 5, 4].

Sequential coalition	Weight tallies	Pivotal player
$\langle P_1, P_2, P_3 \rangle$	5, **10**	P_2
$\langle P_1, P_3, P_2 \rangle$	5, **9**	P_3
$\langle P_2, P_1, P_3 \rangle$	5, **10**	P_1
$\langle P_2, P_3, P_1 \rangle$	5, **9**	P_3
$\langle P_3, P_1, P_2 \rangle$	4, **9**	P_1
$\langle P_3, P_2, P_1 \rangle$	4, **9**	P_2

$SS_1 = 2 \quad SS_2 = 2 \quad SS_3 = 2$

Sh.-Sh. power distribution: $(2/6, 2/6, 2/6) = (1/3, 1/3, 1/3)$

All players have equal power.

Shapley-Shubik Power: Example 3

Example 3: The weighted voting system [10; 5, 5, 4].

Sequential coalition	Weight tallies	Pivotal player
$\langle P_1, P_2, P_3 \rangle$	5, **10**	P_2
$\langle P_1, P_3, P_2 \rangle$	5, 9, **14**	P_2
$\langle P_2, P_1, P_3 \rangle$	5, **10**	P_1
$\langle P_2, P_3, P_1 \rangle$	5, 9, **14**	P_1
$\langle P_3, P_1, P_2 \rangle$	4, 9, **14**	P_2
$\langle P_3, P_2, P_1 \rangle$	4, 9, **14**	P_1

$SS_1 = 3 \quad SS_2 = 3 \quad SS_3 = 0$

Sh.-Sh. power distribution: $(3/6,\ 3/6,\ 0) = (1/2,\ 1/2,\ 0)$

Shapley-Shubik Power: Example 3

Example 3: The weighted voting system [10; 5, 5, 4].

Sequential coalition	**Weight tallies**	**Pivotal player**
$\langle P_1, P_2, P_3 \rangle$	5, **10**	P_2
$\langle P_1, P_3, P_2 \rangle$	5, 9, **14**	P_2
$\langle P_2, P_1, P_3 \rangle$	5, **10**	P_1
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$SS_1 = 3 \quad SS_2 = 3 \quad SS_3 = 0$

Sh.-Sh. power distribution: $(3/6, 3/6, 0) = (1/2, 1/2, \mathbf{0})$

Player $\mathbf{P_3}$ is a dummy.

Shapley-Shubik Power: Example 4

Example 4: The weighted voting system [12; 5, 5, 4].

Sequential coalition	**Weight tallies**	**Pivotal player**
$\langle P_1, P_2, P_3 \rangle$	5, 10, **14**	P_3
$\langle P_1, P_3, P_2 \rangle$	5, 9, **14**	P_2
$\langle P_2, P_1, P_3 \rangle$	5, 10, **14**	P_3
$\langle P_2, P_3, P_1 \rangle$	5, 9, **14**	P_1
$\langle P_3, P_1, P_2 \rangle$	4, 9, **14**	P_2
$\langle P_3, P_2, P_1 \rangle$	4, 9, **14**	P_1

$SS_1 = 2 \quad SS_2 = 2 \quad SS_3 = 2$

Sh.-Sh. power distribution: $(2/6, 2/6, 2/6) = (1/3, 1/3, 1/3)$

Shapley-Shubik Power: Example 4

Example 4: The weighted voting system [12; 5, 5, 4].

Sequential coalition	**Weight tallies**	**Pivotal player**
$\langle P_1,\ P_2,\ P_3\rangle$	5, 10, **14**	P_3
$\langle P_1,\ P_3,\ P_2\rangle$	5, 9, **14**	P_2
$\langle P_2,\ P_1,\ P_3\rangle$	5, 10, **14**	P_3
$\langle P_2,\ P_3,\ P_1\rangle$	5, 9, **14**	P_1
$\langle P_3,\ P_1,\ P_2\rangle$	4, 9, **14**	P_2
$\langle P_3,\ P_2,\ P_1\rangle$	4, 9, **14**	P_1

$SS_1 = 2 \quad SS_2 = 2 \quad SS_3 = 2$

Sh.-Sh. power distribution: $(2/6,\ 2/6,\ 2/6) = (1/3,\ 1/3,\ 1/3)$

All players have equal power.

Shapley-Shubik Power: Examples 2 and 4

Note: The WVS's [8; 5, 5, 4] and [12; 26, 5, 5, 4] have the same Shapley-Shubik power distribution, namely

$$(1/3,\ 1/3,\ 1/3)$$

(see Examples 2 and 4 above).

However, these two WVS's are **not** equivalent to each other.

The WVS [8; 26, 5, 5, 4] requires only two out of three players to agree, while [12; 26, 5, 5, 4] requires unanimity.

On the other hand, **if** two WVS's are equivalent then they must have the same Shapley-Shubik power distribution.

Shapley-Shubik Power: Example 5

Example 5: The weighted voting system [22; 26, 5, 5, 4].

- $4! = 24$ sequential coalitions. Must we write them all out?

Shapley-Shubik Power: Example 5

Example 5: The weighted voting system [22; 26, 5, 5, 4].

- $4! = 24$ sequential coalitions. Must we write them all out?
- Fortunately, no. In **every** sequential coalition, P_1 ("Mom") is the pivotal player, no matter what position she is in.

$SS_1 = 24 \quad SS_2 = SS_3 = SS_4 = 0$

Shapley-Shubik power distribution: (**1**, 0, 0, 0)

Player $\mathbf{P_1}$ is a dictator.

Example: The UN Security Council

The UN Security Council consists of

- 5 permanent members (China, France, Russia, UK, USA)
- 10 rotating members

Passing a motion requires at least nine votes, including all five of the permanent members. (Assume no abstentions.)

What is the Shapley-Shubik power distribution?

(If you thought $\mathbf{2^{15} = 32,768}$ **coalitions** was a lot to consider for Banzhaf power, now it's much worse: there are **15! = 1,307,674,368,000 sequential coalitions**.)

Example: The UN Security Council

Permanent members: each pivotal for 256,657,766,400 s.c.'s
Rotating members: each pivotal for 2,438,553,600 s.c.'s

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Sh.-Sh. power index of each permanent member:

$$\frac{256,657,766,400}{1,307,674,368,000} \approx 0.1963 = \mathbf{19.63\%}$$

Example: The UN Security Council

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Sh.-Sh. power index of each permanent member:

$$\frac{256,657,766,400}{1,307,674,368,000} \approx 0.1963 = \mathbf{19.63\%}$$

Sh.-Sh. power index of each rotating member:

$$\frac{2,438,553,600}{1,307,674,368,000} \approx 0.0019 = \mathbf{0.19\%}$$

Banzhaf vs. Shapley-Shubik

	Permanent	**Rotating**
Banzhaf	16.69%	1.65%
Shapley-Shubik	19.63%	0.19%

- Both power indices confirm that the permanent members have **much more power** than rotating members. (It is moot whether France has 10 times or 100 times the power that Gabon has.)

Banzhaf vs. Shapley-Shubik

So, which is a more accurate measure of power? The Banzhaf power distribution or the Shapley-Shubik power distribution?

Banzhaf vs. Shapley-Shubik

The Banzhaf and Shapley-Shubik power indices give the same answer to the question of which of two players is more powerful:

- If Alice has a higher BPI than Bob, then Alice has a higher SSPI than Bob.
- If Alice has the same BPI as Bob, then Alice has the same SSPI as Bob.
- If Alice has a lower BPI than Bob, then Alice has a lower SSPI than Bob.

Banzhaf vs. Shapley-Shubik

- **Both** the Banzhaf and Shapley-Shubik power indices **recognize dictators correctly**.

  Any dictator has both BPI and SSPI equal to 1 (=100%).

  Conversely, any non-dictator has both BPI and SSPI less than 1.

- **Both** the Banzhaf and Shapley-Shubik power indices **recognize dummies correctly**.

  Any dummy has both BPI and SSPI equal to 0.

  Conversely, any non-dummy has both BPI and SSPI more than 0.

Banzhaf vs. Shapley-Shubik

- **Both** the Banzhaf and Shapley-Shubik power indices **recognize when two players have equal power**.
- **Both** measures of power **yield the same answers for equivalent weighted voting systems.**
- **Neither** can always **distinguish non-equivalent weighted voting systems**.

  For example, neither Banzhaf nor Shapley-Shubik can tell the difference between [8; 5, 5, 4] and [12; 5, 5, 4].

Banzhaf vs. Shapley-Shubik

The upshot: The similarities between the Banzhaf and Shapley-Shubik power distributions are more important than the differences, and both are are good ways to measure power.

- For a large WVS, one distribution may be easier to calculate than the other. There are many fewer coalitions (2^N) than sequential coalitions ($N!$), but on the other hand sequential coalitions are more symmetric.
- Depending on the WVS being studied, it may be more accurate to model coalitions as groups of players that form all at once (Banzhaf), or as groups that players join one at a time (Shapley-Shubik).