The Method of Pairwise Comparisons

Suggestion from a Math 105 student (8/31/11): **Hold a knockout tournament between candidates.**

- This satisfies the Condorcet Criterion! A Condorcet candidate will win all his/her matches, and therefore win the tournament. (Better yet, seeding doesn't matter!)
- **But,** if there is no Condorcet candidate, then it's not clear what will happen.
- Using preference ballots, we can actually hold a **round-robin** tournament instead of a knockout.

The Method of Pairwise Comparisons (§1.5)

The Method of Pairwise Comparisons

Proposed by Marie Jean Antoine Nicolas de Caritat, marquis de Condorcet (1743–1794)

- Compare each two candidates head-to-head.
- Award each candidate one point for each head-to-head victory.
- The candidate with the most points wins.

The Method of Pairwise Comparisons

Number of Voters	**14**	**10**	**8**	**4**	**1**
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

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4th choice	D	A	A	A	A

Compare A to B.

- 14 voters prefer A.
- $10+8+4+1 = 23$ voters prefer B.
- B wins the pairwise comparison and gets 1 point.

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1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Compare C to D:

- 14+10+1 = 25 voters prefer C.
- 8+4 = 12 voters prefer D.
- C wins the pairwise comparison and gets 1 point.

The Method of Pairwise Comparisons

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2nd choice	B	B	C	D	D
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4th choice	D	A	A	A	A

- Compare A to C... A to D... B to C... B to D...

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	A	B	C	D	**Wins**	**Losses**	**Points**
A							
B							
C							
D							

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2nd choice	**B**	**B**	C	D	D
3rd choice	C	D	**B**	C	**B**
4th choice	D	**A**	**A**	**A**	**A**

	A	B	C	D	**Wins**	**Losses**	**Points**
A		14					
B	23						
C							
D							

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3rd choice	C	D	B	C	B
4th choice	D	**A**	**A**	**A**	**A**

	A	B	C	D	**Wins**	**Losses**	**Points**
A		14	14	14			
B	23						
C	23						
D	23						

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2nd choice	**B**	**B**	**C**	D	D
3rd choice	**C**	D	**B**	**C**	**B**
4th choice	D	A	A	A	A

	A	B	C	D	**Wins**	**Losses**	**Points**
A		14	14	14			
B	23		18				
C	23	19					
D	23						

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2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

	A	B	C	D	**Wins**	**Losses**	**Points**
A		14	14	14			
B	23		18	28			
C	23	19		25			
D	23	9	12				

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2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

	A	B	C	D	**Wins**	**Losses**	**Points**
A		14	14	14	—	B,C,D	0
B	23		18	28	A,C	D	2
C	23	19		25	A,B,D	—	3
D	23	9	12		A	B,C	1

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3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

	A	B	C	D	**Wins**	**Losses**	**Points**
A		14	14	14	—	B,C,D	0
B	23		18	28	A,C	D	2
C	23	19		25	A,B,D	—	3
D	23	9	12		A	B,C	1

Winner!

Evaluating the Method of Pairwise Comparisons

- The Method of Pairwise Comparisons satisfies the Majority Criterion.

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- The Method of Pairwise Comparisons satisfies the Condorcet Criterion.

  (A Condorcet candidate will win every pairwise comparison — that's what a Condorcet candidate is!)

Evaluating the Method of Pairwise Comparisons

- The Method of Pairwise Comparisons satisfies the Public-Enemy Criterion.

  (If there is a public enemy, s/he will lose every pairwise comparison.)

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  (Ranking Candidate X higher can only help X in pairwise comparisons.)

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  (Ranking Candidate X higher can only help X in pairwise comparisons.)

Does the Method of Pairwise Comparisons have any drawbacks?

How Many Pairwise Comparisons?

Problem #1: It's somewhat inefficient. How many pairwise comparisons are necessary if there are N candidates?
How many spaces are there in the crosstable?

	A	B	C	D	E
A					
B					
C					
D					
E					

How Many Pairwise Comparisons?

- N^2 squares in crosstable
- N squares on the main diagonal don't count
- Other squares all come in pairs

$$\textbf{Number of comparisons} = \frac{N^2 - N}{2} = \frac{N(N-1)}{2}.$$

Be Careful!

Number of pairwise comparisons with N candidates:

$$\frac{N(N-1)}{2}.$$

Number of points on a Borda count ballot with N candidates:

$$\frac{N(N+1)}{2}.$$

(To remember which is which, work out a small example, like $N = 3$.)

Evaluating the Method of Pairwise Comparisons

Problem #2 (the "rock-paper-scissors problem"):

Ties are very common under the Method of Pairwise Comparisons.

Evaluating the Method of Pairwise Comparisons

Number of voters	**4**	**3**	**6**
1st	A	B	C
2nd	B	C	A
3rd	C	A	B

- The Method of Pairwise Comparisons results in a three-way tie.
- Under any other system we have discussed, C would win.

Comparison of Voting Methods

	Maj	Cond	PE	Mono
Plurality	**Yes**	**No**	**No**	**Yes**
Borda Count	**No**	**No**	**Yes**	**Yes**
PWE	**Yes**	**No**	**Yes**	**No**
Pairwise Comparisons	**Yes**	**Yes**	**Yes**	**Yes**

Maj = Majority; Cond = Condorcet;
PE = Public-Enemy; Mono = Monotonicity

The IIA Criterion

Number of voters	**9**	**11**	**7**	**6**	**3**
1st	A	B	D	C	D
2nd	C	A	B	A	C
3rd	D	C	C	D	B
4th	B	D	A	B	A

(1) Who wins?

The IIA Criterion

Number of voters	**9**	**11**	**7**	**6**	**3**
1st	A	B	D	C	D
2nd	C	A	B	A	C
3rd	D	C	C	D	B
4th	B	D	A	B	A

(2) What happens if D is disqualified?

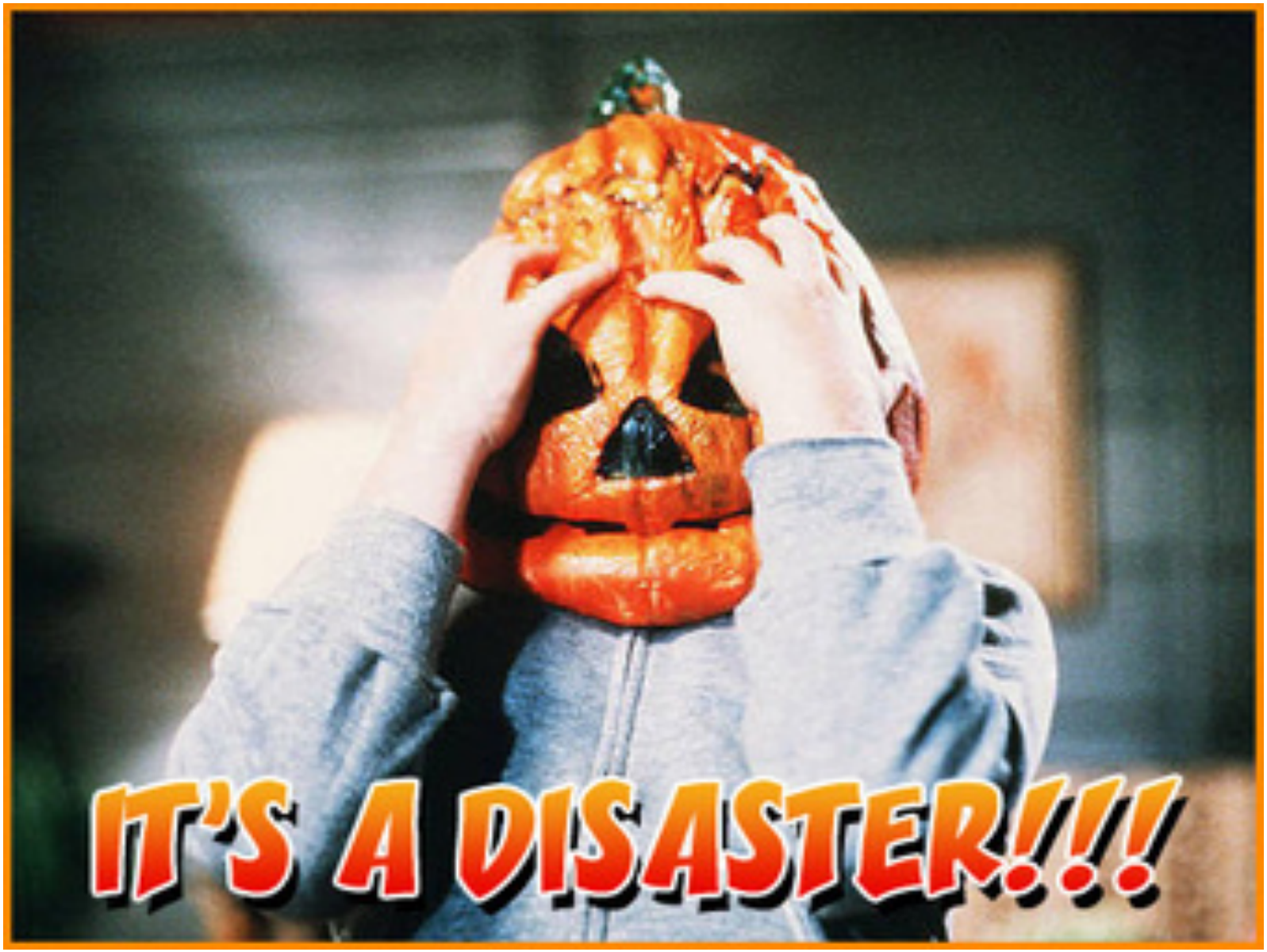
The IIA Criterion

Independence-Of-Irrelevant-Alternatives (IIA) Criterion:

If Candidate A is the winner of an election, and Candidate B is suddenly disqualified, then A should still win the election.

We have just seen that the Method of Pairwise Comparisons violates IIA.

Unfortunately, **none** of the systems we have studied always meet the IIA Criterion!


IT'S A DISASTER!!!

Comparison of Voting Methods

	Maj	Cond	PE	Mono	IIA
Plurality	**Yes**	**No**	**No**	**Yes**	**No**
Borda Count	**No**	**No**	**Yes**	**Yes**	**No**
Plurality-With-Elim.	**Yes**	**No**	**Yes**	**No**	**No**
Pairwise Comparisons	**Yes**	**Yes**	**Yes**	**Yes**	**No**

Maj = Majority; Cond = Condorcet;
PE = Public-Enemy; Mono = Monotonicity;
IIA = Independence of Irrelevant Alternatives

Which Voting System Is Best?

So, which voting system Is best?

There is no purely mathematical answer to this question.

Arrow's Theorem: There is no voting system that always satisfies all four voting criteria – Majority, Condorcet, Monotonicity and IIA.

So, the answer depends which fairness criteria you think are the most important.