The Cover Time of a Bridge Deal
by Jeremy Martin, August 8, 2019

How does the club's new dealing machine work? It starts by generating a random deal (like what you see in a sheet of hand records). You feed it a deck of cards and an empty duplicate board. Then it goes through the cards one by one, using a scanner to figure out what the card is, consulting the deal record to find out which hand the card belongs to, and then stuffing it in the appropriate slot. Contrary to popular belief, the deals really are random; they are not rigged to make all your finesses fail. You can go in the back room at KVBC anytime and watch the machine at work. Grant recently asked me the following question: **When the machine makes a board, on average, how many cards are dealt before all four slots in the board have at least one card in them?**

Well, that's a good question. Clearly the answer has to be at least four and no more than 40 (since if 40 cards have been dealt then only 12 are left). Also, notice that the question includes the word **"average" —** it has to, since the number we're looking for will vary deal to deal. So we are really looking for a **distribution:** a table of the probabilities that the number of cards needed is each of 4, 5, 6, . . . , 39, 40.

We're going to refer to this number a lot, and I don't want to keep saying "the number of cards that have to be dealt before each hand receives at least one card," so for ease of reference I'm going to call it the **cover time** of the deal. (Here we're measuring time by number of cards dealt, and we're asking how long it takes until each slot in the duplicate board is covered by at least one card.) Also, I'll call that critical card the **cover card**, and the hand who receives it is the **cover hand**.

As a warmup, what's the probability that a deal has cover time 4? Equivalently, what is the probability that the first four cards include exactly one card from each of the North, South, East and West hands? Phrasing the question in this way makes it easier to see how to calculate the answer. The number of ways to choose one card from each of those hands is $13^4 = 28561$, while the number of ways[1] to choose four cards out of 52 is $\binom{52}{4} = 270725$. So the probability that a random deal has cover time 4 is

$$\frac{13^4}{\binom{52}{4}} = \frac{28561}{270725} \approx 10.55\%. \tag{1}$$

(Here's an equivalent way of doing the calculation, pointed out by Mickey Imber. For $n = 4$, each card has to go in a new slot. The probabilities that the first, second, third, and fourth cards each land in a previously unoccupied slot are 52/52, 39/51, 26/50, and 13/49. Their product is

$$\frac{52}{52} \times \frac{39}{51} \times \frac{26}{50} \times \frac{13}{49} = 13^4 \times \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49} \tag{2}$$

which is the same number as in (1) — as we mathematicians like to say, the proof is left to the reader.)

This calculation gets harder for cover times greater than 4, because there are more possibilities about how the cards might be distributed and we have to worry about who the cover hand is (which we were able to ignore just now). However, the problem has a number of features that simplify our life:

1. The suits and ranks of the cards don't matter — all we care about is whose hand a card belongs to. The deck might just as well contain 13 cards labelled each of N, S, E, and W; the problem would be exactly the same.

2. The slots are all interchangeable, so we might as well assume that (say) South is the cover hand, run the numbers for this case, and then multiply by 4. In some calculations, you have to be careful when making

[1] In general the number of ways to choose an (unordered) set of k things out of a set of n things is $n!/k!/(n-k)!$. This number is called a *binomial coefficient*, and the symbol for it is $\binom{n}{k}$; you might also see it written as $C(n, k)$ or ${}_nC_k$.

an assumption like this because of overlap between cases. Here, there's no overlap — on every deal, there is exactly one cover hand.

3. We don't care about the order of the cards before the cover card. In other words, if we want to know the probability that South is the cover hand and that the cover time is c, then we just have to determine the probability that (i) each of West, North and East has at least one card among the first $c-1$ cards, but South does not; and (ii) the c^{th} card belongs to South. The probability that (i) happens is[2]

$$\frac{\displaystyle\sum_{(w,n,e)} \binom{13}{w}\binom{13}{n}\binom{13}{e}}{\displaystyle\binom{52}{c-1}}. \tag{3}$$

In other words: "Pick three positive numbers w, n, e that add up to $c-1$. Then pick w, n and e cards from the West, North and East hands respectively. Sum up the number of ways of doing all this, and divide the sum by the number of ways the first $c-1$ cards might be chosen." Given that (i) happens, the probability that (ii) then happens is $13/(53-c)$ (since after $c-1$ cards have been dealt there are $52-(c-1) = 53-c$ left, of which 13 belong to South). Therefore, the probability that the cover time is c and that South is the cover hand is

$$\frac{\displaystyle\sum_{(w,n,e)} \binom{13}{w}\binom{13}{n}\binom{13}{e}}{\displaystyle\binom{52}{c-1}} \times \frac{13}{53-c}. \tag{4}$$

Multiplying by 4 gives what we are looking for: the probability that the cover time is c (with any of the four possible cover hands):

$$P(c) = \frac{\displaystyle 4\sum_{(w,n,e)} \binom{13}{w}\binom{13}{n}\binom{13}{e}}{\displaystyle\binom{52}{c-1}} \times \frac{13}{53-c} \tag{5}$$

which can be simplified a bit to yield

$$\frac{\displaystyle\sum_{(w,n,e)} \binom{13}{w}\binom{13}{n}\binom{13}{e}}{\binom{51}{c-1}}. \tag{6}$$

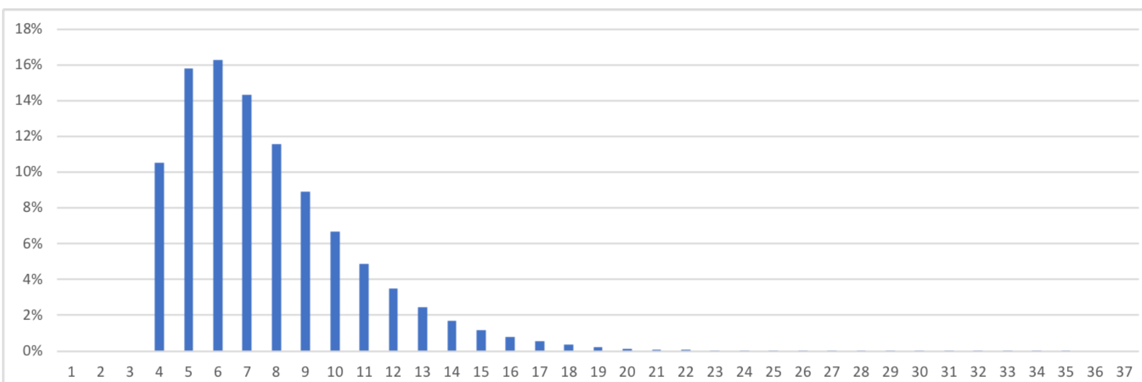
Fortunately, I have a piece of software called Sage that can calculate things like this pretty easily — it will take care of the dirty work not just of calculating huge numbers like $\binom{52}{17}$, but also of figuring out all the possibilities for the numbers w, n and e in terms of a particular c. I'll spare you the actual code, but here are the results:

[2]Here and in the following formulas, the sum ranges over all triples w, n, e of positive integers that add up to $c-1$. Strictly speaking these conditions should be written under the sum itself, but the formula would become very awkward to typeset.

Cover time c	Probability $P(c)$	Cover time c	Probability $P(c)$
1	0%	11	4.85%
2	0%	12	3.47%
3	0%	13	2.45%
4	10.55%	14	1.70%
5	15.82%	15	1.16%
6	16.27%	16	0.79%
7	14.31%	17	0.53%
8	11.59%	18	0.35%
9	8.93%	19	0.22%
10	6.66%	20	0.14%

After $c = 20$, the numbers get very small indeed, less than 0.1% (i.e., less than 1 in 1000), before dropping all the way to zero at $c = 41$ (as explained above). The probability that $c = 40$ is the same as the probability of one hand being dealt a thirteen-card suit (you might want to think about why this is; for a hint, read point #1 above!), which is less than once in a hundred billion deals.

Here's the same data in histogram form:



Some more statistical quantities we can obtain from this distribution:

- The **average** (or **mean**) cover time is about $c = 6.66$. (In other words, if you generate a zillion deals, add up all the cover times, and divide by the number of deals, you will likely get this value.)
- The **median** is $c = 7$. (If you generate a zillion deals and sort them by cover time, the one in the middle will have cover time 7.)
- The **mode** is $c = 6$. (This is the single most likely cover time — the highest column in the histogram.)