A Higher-Dimensional Sandpile Map

AMS Fall Sectional

Alex McDonough

Brown University

Based on arxiv.org/abs/2007.09501

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- This proof will use a geometric construction that gives a periodic tiling of space.
- My goal is for the entire talk to be understandable to a general math audience.

Standard Representative Matrices

Definition

A *standard representative matrix* D is an $(r \times (n+r))$ matrix of the form $\begin{bmatrix} I_r & M \end{bmatrix}$ for some integer matrix M.

$$D = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 0 \end{pmatrix} \end{array}$$

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Note

Any *cell complex* or *orientable arithmetic matroid* satisfying a mild condition can be associated with a unique(ish) standard representative matrix.

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$$m(\{v_1, v_2\}) = 1, m(\{v_1, v_3\}) = 1, m(\{v_2, v_3\}) = 2,$$
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D and $\hat{D}$

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- $\hat{D}$ relates to D in several ways that we will explore on the next slide.

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- If we put D on top of $\hat{D}$, we get an invertible square matrix of the form:

$$\mathcal{D} = \begin{bmatrix} I_r & N \\ -N^T & I_n \end{bmatrix}$$

Representative Matrix Sandpile Group

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Theorem (Duval-Klivans-Martin, 2009)

$$|\mathcal{S}(D)| = \sum_{B \in \mathcal{B}(D)} m(B)^2.$$

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$$|\mathcal{S}(D)| = m(\{v_1, v_2\})^2 + m(\{v_1, v_3\})^2 + m(\{v_2, v_3\})^2 =$$

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^2 + \det\left(\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\right)^2 + \det\left(\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}\right)^2 = 1^2 + 2^2 + 3^2 = 14.$$

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- Recently, I defined a family of meaningful maps $f : \mathcal{S}(D) \to \mathcal{B}(D)$ for any standard representative matrix D such that for every $B \in \mathcal{B}(D)$, we have $|f^{-1}(B)| = m(B)^2$.

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- Recently, I defined a family of meaningful maps $f : \mathcal{S}(D) \to \mathcal{B}(D)$ for any standard representative matrix D such that for every $B \in \mathcal{B}(D)$, we have $|f^{-1}(B)| = m(B)^2$. My goal of this presentation is to share these maps.

Fundamental Parallelepipeds

Definition

The *fundamental parallelepiped* of a square matrix M with column vectors $v_1, \ldots, v_n$ is the set of points:

$$\left\{ \sum_{i=1}^{n} a_i v_i \mid 0 \leq a_i \leq 1 \right\}.$$

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- The polytope $\Pi_\bullet(M)$ is also the *zonotope* or *minkowski sum* of the columns vectors that make up M.
- In order to construct our maps, we associate each basis with the fundamental parallelepiped of a particular matrix.

Basis Parallelepipeds

Let $D = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix} \end{array}$ which means that $\mathcal{D} = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \left(\begin{array}{ccc} 1 & 0 & 3 \\ 0 & 1 & 2 \\ \hline -3 & -2 & 1 \end{array}\right) \end{array}$.

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- For each basis $B \in \mathcal{B}(D)$, we get a parallelepiped $P(B)$ by replacing the first r or last n entries of each column of $\mathcal{D}$ by 0 based on which columns make up B.

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$$P(\{v_1, v_2\}) = \Pi_\bullet\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \quad P(\{v_1, v_3\}) = \Pi_\bullet\left(\begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}\right)$$

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$$T(D) = \Pi_\bullet\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) \bigcup \Pi_\bullet\left(\begin{bmatrix}1 & 0 & 3\\ 0 & 0 & 2\\ 0 & -2 & 0\end{bmatrix}\right) \bigcup \Pi_\bullet\left(\begin{bmatrix}0 & 0 & 3\\ 0 & 1 & 2\\ -3 & 0 & 0\end{bmatrix}\right)$$

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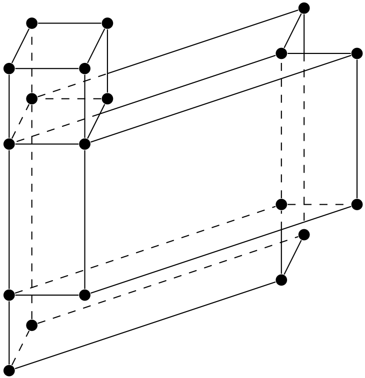

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The parallelepipeds that make up $T(D)$ have non overlapping interiors. Furthermore, the translates of $T(D)$ by integer linear combinations of rows of $\mathcal{D}$ form a non-overlapping tiling of $\mathbb{R}^{r+n}$.

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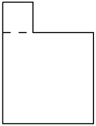

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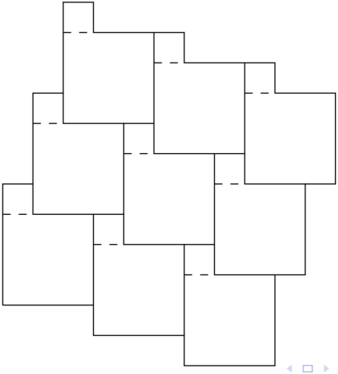

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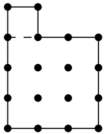

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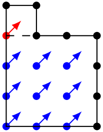

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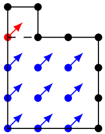

- For our map, we can associate each point $z \in \mathbb{Z}^2$ with a basis B such that $z \in P(B)$.
- To do this, we nudge the points in some generic direction and see where they end up.
- This construction always maps $m(B)^2$ points into each $P(B)$.

Conclusion

Theorem (M. 2020)

For any $(r \times (r+n))$ standard representative matrix D, and any generic direction vector $w \in \mathbb{R}^{r+n}$, we constructed a natural map $f_w : \mathcal{S}(D) \to \mathcal{B}(D)$ such that for every $B \in \mathcal{B}(D)$, we have $|f^{-1}(B)| = m(B)^2$.

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For any $(r \times (r+n))$ standard representative matrix D, and any generic direction vector $w \in \mathbb{R}^{r+n}$, we constructed a natural map $f_w : \mathcal{S}(D) \to \mathcal{B}(D)$ such that for every $B \in \mathcal{B}(D)$, we have $|f^{-1}(B)| = m(B)^2$.

- We did this by first constructing a polytope $P(B)$ for each basis $B \in \mathcal{B}(D)$ and combining them to form $T(D)$, which periodically tiles $\mathbb{R}^{r+n}$ by translations of $\mathcal{D}$.

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- These maps specialize to the maps given by Backman, Baker, and Yuen.

Pretty Pictures

- Because of the structure of $\mathcal{D}$, we can also tile $\mathbb{R}^r$ or $\mathbb{R}^n$ instead of $\mathbb{R}^{r+n}$.

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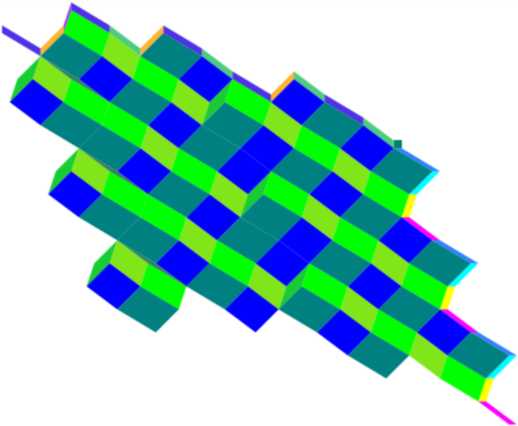

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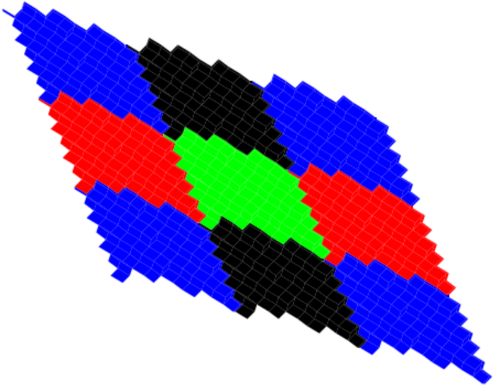

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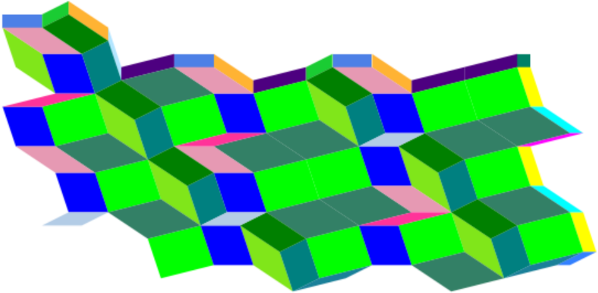

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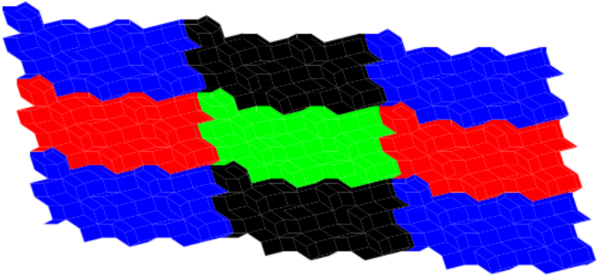

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Thanks For Listening!!!