Whitney Numbers for Poset Cones

AMS Central in El Paso, Texas (Online)

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September 13, 2020

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Table of contents

The Problem

Hyperplane Arrangements

This presentation concerns *cones* $\mathcal{K}$ of arrangements of hyperplanes $\mathcal{A} = \{H_1, \ldots, H_m\}$ in a real vector space $V \cong \mathbb{R}^n$.

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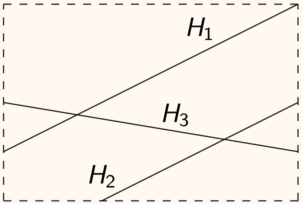
- Every *lower interval*

$$[V, X] := \{Y \in \mathcal{L}(\mathcal{A}) : V \leq Y \leq X\}$$

  of $\mathcal{L}(\mathcal{A})$ forms a *geometric lattice*. In particular, each such $[V, X]$ is a *ranked* poset, with rank function given by the $\operatorname{codim}(X) := n - \dim(X)$.

Example

Here is an arrangement $\mathcal{A} = \{H_1, H_2, H_3\} \subseteq \mathbb{R}^2$ (left) together with the Hasse diagram of its intersection poset $\mathcal{L}(\mathcal{A})$ (right).



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\begin{tikzcd}
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Cones in an Arrangement

Definition (Cone)

A *cone* $\mathcal{K}$ of an arrangement $\mathcal{A}$ is an intersection of half spaces defined by some of the hyperplanes of $\mathcal{A}$.

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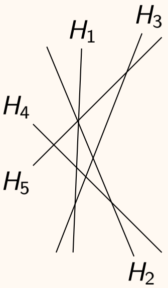

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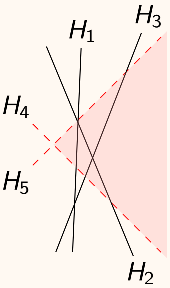

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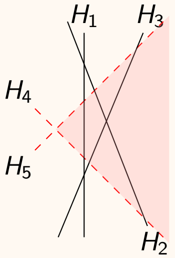

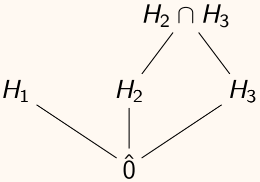

Zaslavsky's Theorem for cones

Theorem (Zaslavsky's Theorem for Cones)

For a cone $\mathcal{K}$ of an arrangement $\mathcal{A}$ with intersection poset $\mathcal{L}^{\text{int}}(\mathcal{K})$, we have

$$\#\mathcal{C}(\mathcal{K}) = \sum_{X \in \mathcal{L}^{\text{int}}(\mathcal{K})} |\mu(V, X)| = \sum_{k=0}^{n} c_k(\mathcal{K})$$

where $\mu(V, X)$ denotes the Möbius function of $\mathcal{L}^{\text{int}}(\mathcal{K})$ and $\{\, c_k(\mathcal{K}) \,\}$ are the *Whitney numbers of the cone* $\mathcal{K}$.

In other words $\#\mathcal{C}(\mathcal{K}) = [\text{Poin}(\mathcal{K}, t)]_{t=1}$, where $\text{Poin}(\mathcal{K}, t)$ is the Poincaré polynomial of $\mathcal{K}$, defined by

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This result is well-known when we take $\mathcal{K}$ to be the full arrangement.

Hyperplane Arrangements

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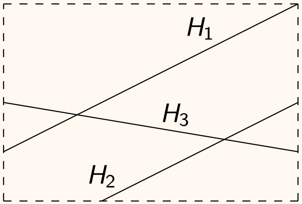

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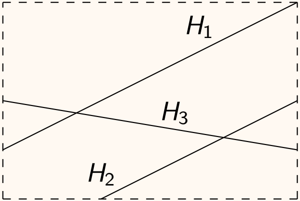

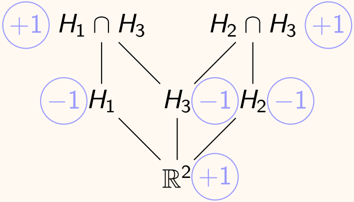

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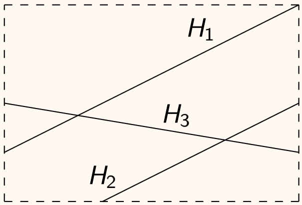

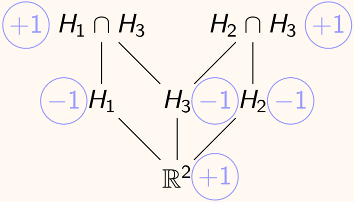

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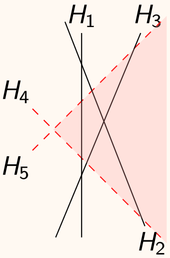

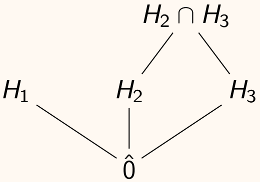

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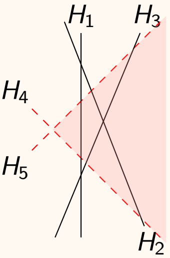

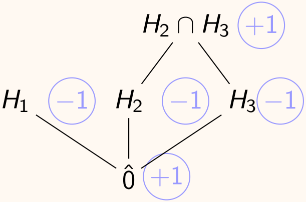

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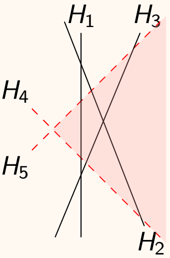

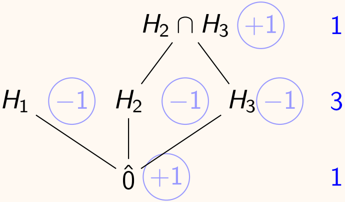

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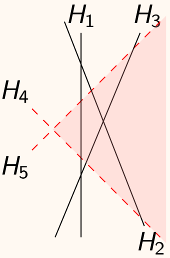

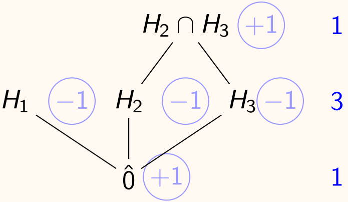

Zaslavsky says: there are $1 + 3 + 1 = 5$ chambers in this cone.

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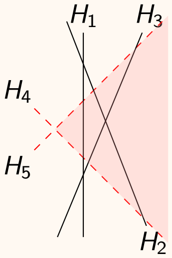

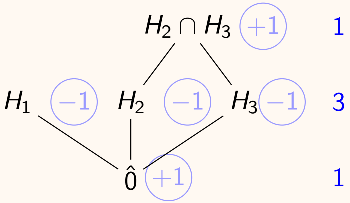

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Goal: Describe the Poincaré polynomial for cones in Type A.

Posets

Cones in Type A

- The braid arrangement $A_{n-1} = \{H_{ij}\}_{1 \leq i < j \leq n}$ is the set of $\binom{n}{2}$ hyperplanes $H_{ij} = \{(x_1, \ldots, x_n) \in V = \mathbb{R}^n \mid x_i - x_j = 0\}$.

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- There is an (easy) bijection between posets on $[n] := \{1, 2, \ldots, n\}$ and cones in the braid arrangement A_{n-1}, given by sending a poset P to the cone

$$\mathcal{K}_P := \{x \in V = \mathbb{R}^n : x_i < x_j \text{ for } i <_P j\}.$$

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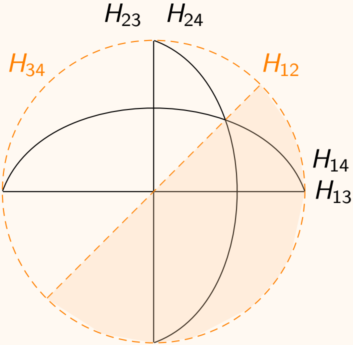
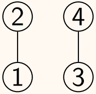
- For any linear order (permutation) on $[n]$, the chamber $\mathcal{K}_\sigma$ lies in the cone $\mathcal{C}(\mathcal{K}_P)$ if and only σ is a linear extension of P.

Example

Consider the cone of A_{4-1} defined by a disjoint union of two chains.



$\leftrightarrow \quad P =$

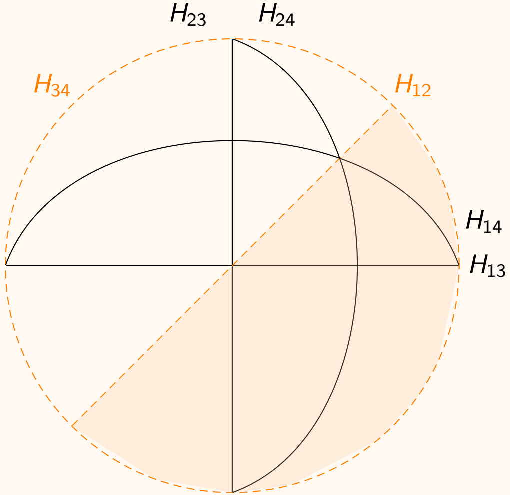


The linear extensions of P are:

$$\mathrm{LinExt}(P) = \{1234, \quad 1324, \quad 1342, \quad 3124, \quad 3142, \quad 3412\}$$

Example(A_{4-1})

We can label the chambers of $\mathcal{K}_P$ by linear extensions of P.



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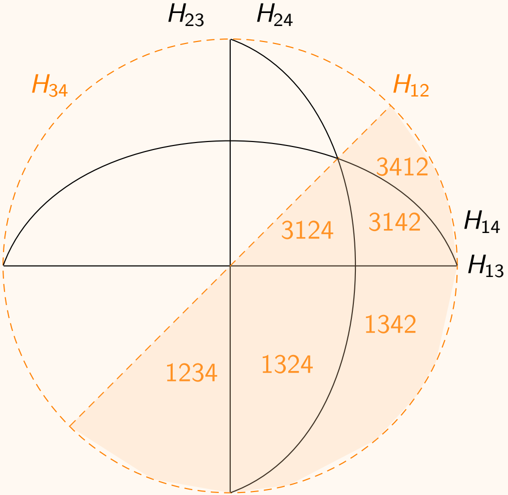

Main Problem for Type A

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Main Problem

Given a poset P on $[n]$, find a statistic $\mathrm{LinExt}(P) \xrightarrow{\mathrm{stat}} \{0, 1, 2, \ldots\}$ interpreting

$$\#\mathrm{LinExt}(P) = \sum_{k \geq 0} c_k(P) = [\mathrm{Poin}(P, t)]_{t=1}$$

as follows:

$$\sum_{\sigma \in \mathrm{LinExt}(P)} t^{\mathrm{stat}(\sigma)} = \sum_{k \geq 0} c_k(P)\, t^k = \mathrm{Poin}(P, t).$$

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Let's motivate this with an example...

Main Problem for Type A: Motivating Example

Example

Let P be an antichain poset on n elements. This corresponds to the full arrangement A_{n-1}. Then

$$1\,(1+t)(1+2t)\cdots(1+(n-1)t) = \sum_{\sigma\in\mathfrak{S}_n} t^{n-\#\text{cycles}(\sigma)}$$

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We'll generalize this example using a notion of *P-transverse permutations*.

P-transverse Permutations

Definition (P-transverse Partition)

Given a poset P on $[n]$, we say that a partition π is *P-transverse* if π corresponds to an intersection interior to the cone $\mathcal{K}_P$.

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Definition (P-transverse Permutation)

Given a poset P on $[n]$, we say that a permutation σ is *P-transverse* if the set partition obtained by forgetting the order within the cycles is P-transverse.

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Example

Let P be the poset on $[4]$ with $1 < 2$ and $3 < 4$ and no other relations. Then the P-transverse permutations are

$$(),\ (13),\ (14),\ (23),\ (24),\ (13)(24).$$

Main Problem for Type A, rephrased

Main Problem for Type A

Given a poset P on $[n]$, Zaslavsky's theorem implies that

$$\#\mathrm{LinExt}(P) = \#\left(P - \text{transverse permutations}\right).$$

Give a combinatorial bijection ψ between these two sets such that

$$\sum_{\sigma \in \mathrm{LinExt}(P)} t^{n - \mathrm{cycles}(\psi(\sigma))} = \sum_{k \geq 0} c_k(P)\, t^k = \mathrm{Poin}(P, t).$$

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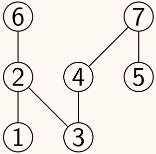
We have such a map! Let's give an example of how it works.

Example $\psi : \textbf{LinExt}(P) \to \mathfrak{S}^{\pitchfork}(P)$

Example

Take $\sigma = 1325476 \in \text{LinExt}(P)$ where

$P =$ 

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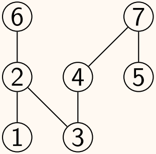
$P =$ 

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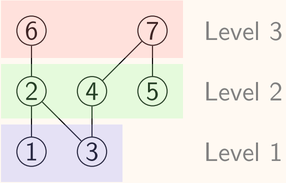
$P =$ 

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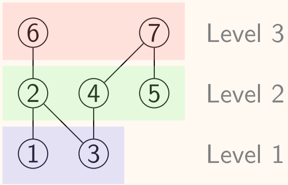

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$$\sigma = 1325476.$$

Say that x in Level k is *essential* if it covers an element of Level $k-1$. We denote the essential elements with an overline.

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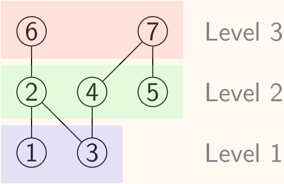
Take $\sigma = 1325476 \in \text{LinExt}(P)$ where

$P =$ 

To compute $\psi(\sigma)$, we cut σ (greedily) into strings which are antichains of P. Here

$$\sigma = \overline{1}\,\overline{3}\,\overline{2}\,5\,\overline{4}\,\overline{7}\,\overline{6}$$

Say that x in Level k is *essential* if it covers an element of Level $k-1$. We denote the essential elements with an overline.

Example $\psi : \mathbf{LinExt}(P) \to \mathfrak{S}^{\pitchfork}(P)$

Example

We have $\sigma = \overline{13}\,\overline{2}5\overline{4}\,\overline{76}$.
Within each level (color block), put a left parenthesis left of each left-to-right maxmimum among the essential elements:

$$\sigma = (\overline{1}(\overline{3}(\overline{2}5(\overline{4}(\overline{76}$$

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Adding in the right parenthesis: $\sigma = (\overline{1})(\overline{3})(\overline{2}5)(\overline{4})(\overline{7}\overline{6})$ Removing the decoration gives

$$\psi(\sigma) = (1)(3)(25)(4)(76)$$

The Theorem

Theorem

Given a poset P on $[n]$, not only does ψ give a bijection, but

$$\sum_{\sigma \in \mathrm{LinExt}(P)} t^{n-\mathrm{cycles}(\psi(\sigma))} = \sum_{k \geq 0} c_k(P)\, t^k = \mathrm{Poin}(P, t).$$

Thanks!

References

[1] Richard Stanley. An Introduction to Hyperplane Arrangements. *Geometric Combinatorics IAS/Park City Mathematics Series*, pages 389–496, 2007.

[2] Thomas Zaslavsky. A Combinatorial Analysis of Topological Dissections. *Advances in Mathematics*, 25(3):267–285, 1977.